Numerical solution of Lord-Shulman thermopiezoelectricity dynamical problem

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Abstract

We consider a generalized Lord-Shulman model of linear thermopiezoelectricity. The corresponding variational problem is formulated based on the initial boundary value problem. Then the variational problem is semi-discretized in space using Galerkin method with finite element method approximations. For complete discretization of the problem, the implicit one-step recurrent scheme for time integration is constructed. The numerical experiment is performed and its results are compared with the ones obtained by the other researchers.

Keywords: generalized thermopiezoelectricity, Lord-Shulman model, PZT-4 ceramics, variational problem, Galerkin method, finite element method, one-step recurrent scheme

1. Introduction

A lot of non-stationary thermoelasticity problems for piezoelectric materials can be satisfactorily solved by applying generalized models of thermopiezoelectricity, among them Lord-Shulman model (LS-thermopiezoelectricity), see Ref. [2].

In this paper we present a numerical scheme for LS-thermopiezoelectricity problem based on finite element approximations in space and implicit one-step recurrent scheme for time integration. Relying on our recent results on well-posedness of variational formulation of LS-thermopiezoelectricity problem, see Ref. [6], we can prove that the numerical scheme is unconditionally stable and has the second order of convergence.

2. Problem statement

The theory of thermopiezoelectricity describes the coupled interaction of mechanical, electrical and thermal fields in piezoelectric materials.

Suppose the piezoelectric specimen occupies a bounded domain Ω in Euclidean space $R^d, d = 1, 2,$ or 3 with Lipschitz boundary Γ. According to the classic theory, we need to find elastic displacement vector $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, electric potential $p = p(\mathbf{x}, t)$ and temperature increment $\theta = \theta(\mathbf{x}, t)$, which satisfy the system of coupled equations of motion, Maxwell's equations in electrostatic approximation and heat conduction equation. The system is complemented by initial and boundary conditions.

In Lord-Shulman model of thermopiezoelectricity, see Ref. [3], instead of standard Fourier's law, the Maxwell-Cattaneo equation is used:

$$\tau q_i' + q_i = -\lambda_{ij}\theta_{,j}. \tag{1}$$

Here $\mathbf{q} = \{q_i\}_{i=1}^d$ is a heat flux vector, prime $'$ means time derivative, λ_{ij} is a tensor of heat conductivity coefficients. The parameter $\tau > 0$ is so-called "relaxation time". This modification ensures finite speeds of heat wave propagation. Like in Ref. [1], we treat $\mathbf{q} = \mathbf{q}(\mathbf{x}, t)$ as an additional independent variable.

3. Variational problem statement

We introduce the spaces of admissible displacements V, electric potentials X, temperature increments Y, heat fluxes Z and notations $\Phi = V \times X \times Y \times Z$, $H = [L^2(\Omega)]^d$. After applying the principle of virtual works to the initial boundary value problem of LS-thermopiezoelectricity, we obtain the following variational problem: given $\psi_0 = (\mathbf{u_0}, p_0, \theta_0, \mathbf{q_0}) \in \Phi$, $\mathbf{v_0} \in H$ and $(l, r, \mu) \in L^2(0, T; \Phi')$; find $\psi = (\mathbf{u}, p, \theta, \mathbf{q}) \in L^2(0, T; \Phi)$ such that

$$\left\{\begin{array}{llll} m(\mathbf{u}''(t),\mathbf{v}) & +a(\mathbf{u}'(t),\mathbf{v}) & +c(\mathbf{u}(t),\mathbf{v}) - & \\ -e(p(t),\mathbf{v}) & -\gamma(\theta(t),\mathbf{v}) & =< l(t),\mathbf{v} >, & \\ & & & \\ \chi(p'(t),\xi) & +e(\xi,\mathbf{u}'(t)) & +z(p(t),\xi) + & \\ & +\pi(\theta'(t),\xi) & =< r(t),\xi >, & \\ & & & \\ s(\theta'(t),\eta) & +\pi(\eta,p'(t)) & +\gamma(\eta,\mathbf{u}'(t)) - & \\ & -g(\mathbf{q}(t),\eta) & =< \mu(t),\eta >, & \\ & & & \\ \tau b(\mathbf{q}'(t),\zeta) & +b(\mathbf{q}(t),\zeta) & +g(\zeta,\theta(t)) = 0 & \\ & & \forall\, t \in (0,T], & \\ & & & \\ \multicolumn{4}{l}{b(\mathbf{q}(0)-\mathbf{q_0},\zeta) = 0 \;\; c(\mathbf{u}(0)-\mathbf{u_0},\mathbf{v}) = 0,} \\ \multicolumn{4}{l}{\chi(p(0)-p_0,\xi) = 0, \;\; s(\theta(0)-\theta_0,\eta) = 0,} \\ \multicolumn{4}{l}{m(\mathbf{u}'(0)-\mathbf{v_0},\mathbf{v}) = 0, \; \forall\phi = (\mathbf{v},\xi,\eta,\zeta) \in \Phi.} \end{array}\right. \tag{2}$$

Here linear forms $< l(t), \cdot >, < r(t), \cdot >, < \mu(t), \cdot >$ represent the mechanical, electric and heat loadings correspondingly. Bilinear form $m(\mathbf{u}, \mathbf{u})$ defines kinetic energy of piezoelectric, $a(\mathbf{u}', \mathbf{u}')$ is responsible for viscosity effect, $c(\mathbf{u}, \mathbf{u})$ defines potential energy, $e(p, \mathbf{v})$ determines interaction between electric and mechanical fields, $\gamma(\eta, \mathbf{u}'(t))$ describes interaction between thermal and mechanical fields, $\chi(p, p)$ defines electric energy, $z(p, p)$ determines electric energy losses, $\pi(\theta, \xi)$ describes the interaction between electric and heat fields, $s(\theta, \theta)$ determines heat energy, for details see Ref. [5]. The definitions of bilinear forms $b(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are shown below:

$$\begin{aligned} b(q,\zeta) &= \int\limits_{\Omega} b_{ij} q_i \zeta_j dx, \\ g(\zeta,\eta) &= \int\limits_{\Omega} T_0^{-1} \zeta_k \eta_{,k} dx \quad \forall \eta \in Y,\ \forall q,\zeta \in Z. \end{aligned} \tag{3}$$

Here b_{ij} are converse coefficients of heat conductivity, T_0 is the initial temperature of the piezoelectric specimen, $\eta_{,k}$ means spatial derivative by x_k. In our recent paper Ref. [6] we proved the well-posedness of this variational problem.

4. Implicit one-step recurrent scheme

For discretization in space we utilize Galerkin method and finite element method for choosing the bases of finite-dimensional subspaces. For discretization in time, we use our developed technique similar to the one described in our previous works (it is actually a combination of Newmark and Crank-Nicolson schemes), see Ref. [4]. As a result, we obtain the following numerical scheme:

$$\begin{cases} \textit{given } \Delta t > 0, \tau > 0,\ 1 \geq \beta \geq \gamma \geq 0, \\ (\dot{U}^j, U^j, P^j, \Theta^j, Q^j); \\ \textit{find vector } \ (\ddot{U}^{j+\frac{1}{2}}, \dot{P}^{j+\frac{1}{2}}, \dot{\Theta}^{j+\frac{1}{2}}, \dot{Q}^{j+\frac{1}{2}}) \textit{ such that} \\ \begin{pmatrix} \tilde{M} & -\Delta t\gamma E^T & -\Delta t\gamma Y^T & 0 \\ \Delta t\gamma E & \tilde{X} & \Pi^T & 0 \\ \Delta t\gamma Y & \Pi & S & -\Delta t\gamma G^T \\ 0 & 0 & \Delta t\gamma G & \tilde{B} \end{pmatrix} \times \\ \times \begin{pmatrix} \ddot{U}^{j+\frac{1}{2}} \\ \dot{P}^{j+\frac{1}{2}} \\ \dot{\Theta}^{j+\frac{1}{2}} \\ \dot{Q}^{j+\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \tilde{L} \\ R_{j+1/2} - E\dot{U}^j - ZP^j \\ F_{j+1/2} - Y\dot{U}^j + G^T Q^j \\ -G\Theta^j - BQ^j \end{pmatrix} \end{cases}, \tag{4}$$

where $\tilde{M} = M + \Delta t\gamma A + \frac{1}{2}\Delta t^2 \beta C$, $\tilde{X} = X + \Delta t\gamma Z$, $\tilde{B} = \tau B + \Delta t\gamma B$, $\tilde{L} = L_{j+1/2} - A\dot{U}^j - CU^j - \Delta t\gamma C\dot{U}^j + E^T P^j + Y^T \Theta^j$. Variables $\ddot{U}^{j+\frac{1}{2}}, \dot{P}^{j+\frac{1}{2}}, \dot{\Theta}^{j+\frac{1}{2}}, \dot{Q}^{j+\frac{1}{2}}$ define the elastic acceleration, time rate of electric potential, time rate of temperature increment and time rate of heat flux propagation correspondingly, which are assumed to be constants on the time interval $[t_j, t_{j+1}]$. The nodal values of the solutions at time moment t_{j+1} can be then calculated using the following expressions:

$$\begin{aligned} &U^{j+1} = U^j + \Delta t \dot{U}^j + \frac{1}{2}\Delta t^2 \ddot{U}^{j+\frac{1}{2}}, \\ &P^{j+1} = P^j + \Delta t \dot{P}^{j+\frac{1}{2}},\ \Theta^{j+1} = \Theta^j + \Delta t \dot{\Theta}^{j+\frac{1}{2}}, \\ &Q^{j+1} = Q^j + \Delta t \dot{Q}^{j+\frac{1}{2}}, j = 0, 1, ..., N_T - 1. \end{aligned} \tag{5}$$

The values $\dot{U}^0, U^0, P^0, \Theta^0$, which are necessary for starting calculations by one-step recurrent scheme Eqn (4)-(5), are easily obtained from the initial conditions of the problem. The aforementioned numerical scheme is unconditionally stable and has the second order of convergence.

5. Numerical experiment

We consider a piezoelectric bar of length $L = 10^{-8} m$ made of PZT-4 ceramics, Ref. [7]. The behavior of the bar is examined during the time interval $[0, T]$, $T = 11.2 \cdot 10^{-12} s$. The boundary conditions for temperature increment θ are supposed to be as follows:

$$\begin{aligned} \theta(0,t) = \theta_c &\begin{cases} \frac{t}{t_p}, 0 \leq t \leq t_p \\ 1,\ t_p \leq t \leq T \end{cases}, \theta_c = 293K, \\ \theta(L,t) &= 0,\ 0 \leq t \leq T \end{aligned} \tag{6}$$

where $t_p = 10^{-12} s$. The boundary conditions for mechanical and electric fields are pure Neumann conditions. The initial disturbances of elastic displacement u, its velocity u', electric potential p and heat flux q are taken to be zeros. Thus, our experiment reproduces the one described by Sumi and Ashida in Ref. [7]. They used method of characteristics for solving the problem, we solve the problem using our one-step recurrent scheme and compare the obtained results with theirs.

For discretization in space we divide the interval $[0, L]$ into $N = 512$ finite elements with piecewise quadratic approximation. For time discretization we divide uniformly the time interval $[0, T]$ into $N_T = 1200$ subintervals. The parameters of the one-step recurrent scheme are taken $\gamma = 0.5$ and $\beta = 0.6$.

Figure 1 shows the time variations of the temperature increment θ for values of relaxation time $\tau = 10^{-11}, 10^{-12}, 10^{-14} s$ respectively. The graphics indicate the significant influence of the relaxation time τ on the behavior of piezoelectric. Also, our results for the temperature agree with those presented in Ref. [7], both qualitatively and quantitatively.

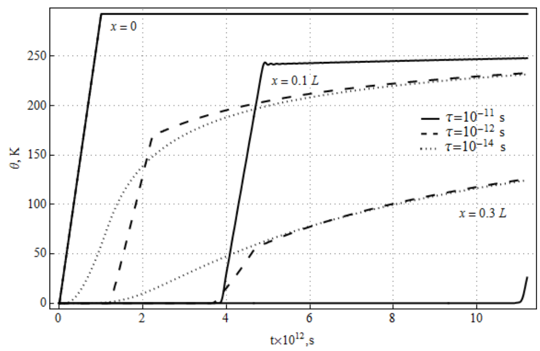

Figure 1: Time variations of temperature increment θ at positions $x = 0$, $x = 0.1L$ and $x = 0.3L$ for the PZT-4 bar for values of relaxation time $\tau = 10^{-11}, 10^{-12}, 10^{-14}$s.

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