# ADVANCED TOPICS IN FINITE ELEMENT METHOD

## Introduction to FEM

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# INTRODUCTION TO FEM

- We will discuss basic concepts and algorithms of the finite element method. I will also include necessary information regarding solid mechanics. As I have written in the introduction, I assume that the reader knows basic issues of mechanics of materials and the theory of elasticity, therefore the information here will be only a short survey and an introduction to the matrix notation.

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# THE CONCEPTS OF FEM

- The beginnings of FEM is the '20s and '30s of the 20th century
- G. B. Maney and H. Cross in the USA and A. Ostenfeld in the Netherlands make use of findings presented in papers written by J. C. Maxwell, A. Castiliano and O. Mohr proposing a new method for solving structural mechanics problems which is now known as the displacement method.

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# THE CONCEPTS OF FEM

- In the middle of the 20th century J. Argyris, P. C. Pattan, S. Kelsey, M. Turner, R. Clough et al. accomplished the generalisation of this method.
- In the '60s and '70s FEM was improved thanks to the publications by O. C. Zienkiewicz, Y. K. Cheung and R. L. Taylor.
- Thus it has become a contemporary tool used for solving different issues.

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# THE CONCEPTS OF FEM

- The basic idea of FEM is to search for a solution to a complex problem by replacing it with a simpler and similar one.
- It leads to the discovery of an approximate solution, the precision of which depends on the assumed approximation methods.
- In mechanics problems, a solution generally consists of determining displacements, strains and stresses in a continuum.

The way of determining an approximate solution in FEM:

- Separation into a number of finite elements (discretization).
- The elements are assumed to be interconnected at a discrete number of nodal point. The displacements of these nodal points will be the basic unknown.

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# THE CONCEPTS OF FEM

- A set of functions is chosen to define the state of displacement within each finite element.
- The displacement functions now define uniquely the state of strain within an element in terms of the nodal displacements.
- Forces concentrated at the nodes are determined.
- The relationship between nodal forces and displacements is described by the element stiffness matrix.
- A set of equilibrium equations is written for all nodes, hence the problem becomes one of solving a set of algebraical equations.

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# LINEAR MODEL OF A STRUCTURE

## Consequences of linear differential equations:

- Linear displacements and angles of rotation are considerably smaller than the characteristic dimension of a structure.
- Strains are small. It enables the relationship between strains and displacements to be expressed with the help of linear equations.
- The material is linear elastic which means that it satisfies Hook's law.

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# THE CONCEPTS OF FEM

- It requires solving many problems (selection of shape functions and discrete systems) which requires some choosing a structure model (elastic, plastic, frame, plate etc.) and a discrete method requires considerable experience.

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# STRESSES

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}$$

- $\sigma_x, \sigma_y, \sigma_z$ - components of direct stress,
- $\tau_{xy}, \tau_{xz}, \tau_{yz}$ - components of shear stress,
- the stress tensor is symmertical $(\tau_{xy}=\tau_{yx}, \tau_{xz}=\tau_{zx}, \tau_{yz}=\tau_{zy})$,

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# STRAINS

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

- $\varepsilon_x$, $\varepsilon_y$, $\varepsilon_z$ are the components of direct strain (unit elongation),
- $\gamma_{xy}$, $\gamma_{xz}$, $\gamma_{yz}$, the components of shear strain (they are the angles of the non-dilatation strain),
- $u_x$, $u_y$, $u_z$ are the components of the displacement vector in the Cartesian coordinate system

$$\varepsilon_x = \frac{\partial u_x}{\partial x}, \quad \varepsilon_y = \frac{\partial u_y}{\partial y}, \quad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

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# WORK OF THE INTERNAL FORCES

- We simplify the calculation of the internal work if we take the components of the strain vector $\gamma_{ij}$ (the angles of the volumetric strain) instead of usual tensor definitions:

$$2W = \int_{\mathcal{V}} \boldsymbol{\sigma}^{\mathsf{T}} \boldsymbol{\varepsilon} \, d\,\mathcal{V} = \int_{\mathcal{V}} \boldsymbol{\varepsilon}^{\mathsf{T}} \mathbf{D} \boldsymbol{\varepsilon} \, d\,\mathcal{V}$$

where: $\mathcal{V}$ – volume of a body

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# CONSTITUTIVE EQUATIONS

$$\boldsymbol{\sigma} = \mathbf{D}\cdot\boldsymbol{\varepsilon}$$

$$\boldsymbol{\varepsilon} = \mathbf{D}^{-1}\cdot\boldsymbol{\sigma}$$

**D -** is the square matrix with dimensions 6×6 containing the material constants.

$\lambda$ and $\mu$ are the Lamé constants,

$\mu$ is also notted by $G$ (Kirchoff's modulus)

$$\mathbf{D} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)}$$

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# CONSTITUTIVE EQUATIONS

- The inverse matrix $\mathbf{D}^{-1}$ with the material constants has an unusually simple structure which is best shown by means of the constants $E, \nu$ :

$$\mathbf{D}^{-1} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}$$

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# PLANE STRESS

- In two-dimensional problems of thin plates, the following simplification of the assumption is:

  $\boldsymbol{\sigma} = \mathbf{D} \cdot \boldsymbol{\varepsilon}$ leads to:

$$\varepsilon_z = -\frac{\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right) \quad \gamma_{zx} = 0 \quad \gamma_{zy} = 0$$

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# PLANE STRESS

- In plane stress, the dimensions of the stress and strain vectors and the matrix of the material constants are reduced by half and thus:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \qquad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

$$\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \qquad \mathbf{D}^{-1} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}$$

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# PLANE STRAIN

- In problems regarding deformations of massive buildings, the plane strain criterion is often found and it is expressed by the equations:

$$\varepsilon_z = 0, \quad \gamma_{zx} = 0, \quad \gamma_{zy} = 0$$

- Knowing, that $\boldsymbol{\varepsilon} = \mathbf{D}^{-1} \cdot \boldsymbol{\sigma}$ we get the ***plane strain:***

$$\sigma_z = \nu(\sigma_x + \sigma_y), \quad \tau_{zx} = 0, \quad \tau_{zy} = 0$$

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# PLANE STRAIN

- The relationship between the reduced stress and strain vectors leads to:

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

$$\mathbf{D}^{-1} = \frac{1-\nu^2}{E} \begin{bmatrix} 1 & \frac{-\nu}{1-\nu} & 0 \\ \frac{-\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{2}{1-\nu} \end{bmatrix}$$

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# EQUILIBRIUM EQUATIONS

- The condition of equilibrium for a fixed body is satisfied when the following six equations called equilibrium equations take place:

$$\sum_{i=1}^{n} \mathbf{P}_i = \mathbf{0}\,, \quad \sum_{i=1}^{n} \mathbf{M}_i = \mathbf{0}, \quad \text{which is:}$$

$$\sum_{i=1}^{n} P_{Xi} = 0 \qquad \sum_{i=1}^{n} P_{Yi} = 0 \qquad \sum_{i=1}^{n} P_{Zi} = 0$$

$$\sum_{i=1}^{n} M_{Xi} = 0 \qquad \sum_{i=1}^{n} M_{Yi} = 0 \qquad \sum_{i=1}^{n} M_{Zi} = 0$$

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# EQUILIBRIUM EQUATIONS

- $P_{Xi}$, $P_{Yi}$, $P_{Zi}$ are the components of the force $\boldsymbol{P}$
- $M_{Xi}$, $M_{Yi}$, $M_{Zi}$ are the moments of this force in relation to the axes of a coordinate system and $n$ is the number of forces.
- When a set of forces is contained in, for example, the plane $XY$, then previous equations are reduced:

$$\sum_{i=1}^{n} P_{Xi} = 0 \qquad \sum_{i=1}^{n} P_{Yi} = 0 \qquad \sum_{i=1}^{n} M_{Zi} = 0$$

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# THE PRINCIPLE OF VIRTUAL WORK

- Equilibrium equation define conditions for a set of forces acting on a rigid body. In the case of an elastic body we have to determine conditions for external forces.
- This can be done by using the principle of virtual work:

$$\sum_{i=1}^{n} \mathbf{P}_i \cdot \overline{\mathbf{u}}_i = E_\sigma$$

$\overline{\mathbf{u}}_i$ is the vector of the virtual displacement at the point $i$, the dot means the scalar product of the force vector $\boldsymbol{P}_i$ and the displacement vector $\overline{\mathbf{u}}_i$

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# THE PRINCIPLE OF VIRTUAL WORK

- $E_\sigma$ - is the potential energy of internal forces:

$$E_\sigma = \int_{\mathcal{V}} \boldsymbol{\sigma}^{\mathsf{T}} \boldsymbol{\varepsilon}\, d\,\mathcal{V} = \int_{\mathcal{V}} \boldsymbol{\varepsilon}^{\mathsf{T}} \boldsymbol{\sigma}\, d\,\mathcal{V}$$

- $\boldsymbol{\varepsilon}$ denotes the strain vector which results from the virtual displacement $\overline{\mathbf{u}}_i$ .

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# THE PRINCIPLE OF VIRTUAL WORK

The virtual displacement must satisfy the following conditions:

- it should be independent of forces acting on a solid,
- it should be consistent with the constraints so that it is kinematically allowable,
- it should be independent of time.

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# CLAPEYRON'S THEOREM

- Changing virtual displacements into the real ones we obtain:

$$\sum_{i=1}^{n} \mathbf{P}_i \cdot \mathbf{u}_i = \int_{\mathcal{V}} \boldsymbol{\sigma}^{\mathsf{T}} \boldsymbol{\varepsilon} \, d\mathcal{V} = \int_{\mathcal{V}} \boldsymbol{\varepsilon}^{\mathsf{T}} \boldsymbol{\sigma} \, d\mathcal{V}$$

Clapeyron's theorem:

- for the elastic body in equilibrium, the work of external forces is equal to the potential energy of internal forces (elastic energy).

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# CLAPEYRON'S THEOREM

The elastic body (Clapeyton' body) conditions:

- material of which the body is composed reacts according to Hook's law,
- body does not possess the boundary conditions which depend on the deformation of a structure,
- body temperature is constant,
- there are no initial stresses and strains.

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# THE BETTI THEOREM

$$\sum_{i=1}^{n} \mathbf{P}_i \cdot \bar{\mathbf{u}}_i = E_\sigma \qquad \text{and} \quad \boldsymbol{\sigma} = \mathbf{D} \cdot \boldsymbol{\varepsilon} \quad \text{gives:}$$

$$\sum_{i=1}^{n} \mathbf{P}_i \cdot \bar{\mathbf{u}}_i = \int_{\mathcal{V}} \boldsymbol{\sigma}^{\mathsf{T}} \boldsymbol{\varepsilon} \, d\mathcal{V} = \int_{\mathcal{V}} (\mathbf{D}\boldsymbol{\varepsilon})^{\mathsf{T}} \boldsymbol{\varepsilon} \, d\mathcal{V} = \int_{\mathcal{V}} \boldsymbol{\varepsilon}^{\mathsf{T}} \mathbf{D} \boldsymbol{\varepsilon} \, d\mathcal{V}$$

- We attach virtual loads (a set of forces $\bar{\mathbf{P}}_j$) acting at the same nodes as the actual loads, but of a different value and direction. The work done by these forces is equal to:

$$\sum_{i=1}^{n} \bar{\mathbf{P}}_j \cdot \mathbf{u}_j = \int_{\mathcal{V}} \boldsymbol{\sigma}^{\mathsf{T}} \boldsymbol{\varepsilon} \, d\mathcal{V} = \int_{\mathcal{V}} (\mathbf{D}\boldsymbol{\varepsilon})^{\mathsf{T}} \boldsymbol{\varepsilon} \, d\mathcal{V} = \int_{\mathcal{V}} \boldsymbol{\varepsilon}^{\mathsf{T}} \mathbf{D} \boldsymbol{\varepsilon} \, d\mathcal{V}$$

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# THE BETTI THEOREM

- We obtain the equation which expresses the reciprocal theorem of work formulated by E. Betti:

$$\sum_{i=1}^{n} \mathbf{P}_i \cdot \bar{\mathbf{u}}_i = \sum_{i=1}^{n} \bar{\mathbf{P}}_i \cdot \mathbf{u}_i$$

„ *The set of forces* $\mathbf{P}_i$ *does the same work at the displacements induced by the set of forces* $\mathbf{P}_j$ *as the set of forces* $\mathbf{P}_j$ *does at displacements induced by forces* $\mathbf{P}_i$ ”

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# THE BETTI THEOREM

- If we bring down both sets of forces to single unit forces acting at the point $a$, we obtain:

$$\mathbf{1}_a \cdot \bar{\mathbf{u}}_a = \bar{\mathbf{1}}_a \cdot \mathbf{u}_a$$

- This relationship is called the reciprocal theorem of displacements .

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# FEM ALGORITHM

## *A. Discretization*

- The division of a structure into finite elements. Frameworks are divided into segments. 2D surfaces are divided into triangular and/or quadrilateral elements. Solids - tetrahedral and hexahedronal elements.
- Save points of elements contacts, coordinates of the nodes and the manner of connection between nodes and elements.

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# FEM ALGORITHM

## *B. Calculation of element stiffness matrices*

- On the basis of material properties and topological data given in the first stage matrices expressing relationships between nodal forces and nodal displacements of an element are formed.

## *C. Aggregation of a global stiffness matrix*

- Now element stiffness matrices are divided into blocks which merge into a global stiffness matrix for which the information about construction topology is used. Modifications taking into consideration boundary conditions are often introduced into the global matrix at that stage.

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# FEM ALGORITHM

## *D. Construction of a global loads vector*

- Here we calculate load vectors of elements which, after being divided into blocks, are inserted into the global vector of nodal loads. When the global vector is built, then its components should be modified with regard to boundary conditions.

## *E. Solution of a set of equations*

- At this stage, a set of linear equations will be solved. In effect, we will obtain the nodal displacements of a structure.

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# FEM ALGORITHM

## *F. Calculation of internal forces and reactions*

- If we obtain displacements, we can then calculate strains, stresses and internal forces in a structure. After having calculated element nodal forces, reactions at constraints (supports) of the construction can also be calculated.
- FEM algorithm

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# FEM ALGORITHM

## Preprocessor:

- The first stage (A) complemented by defining material properties and describing construction loads.
- Contemporary preprocessors are usually graphic programmes equipped with tools simplifying the generation of element meshes.

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# FEM ALGORITHM

**Processor:**

- Stages (B), (C), (D), and (E).
- The processor often deals with a suitable arrangement of equations in order to reduce the amount of memory for the storage of the stiffness matrix and to accelerate the process of solving systems of equations.

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# FEM ALGORITHM

## Postprocessor:

- The sixth stage (F) complemented by graphical output.
- Contemporary FEM systems are equipped with a graphic postprocessor producing colour maps of stresses, displacements and other parameters which simplify analyses.

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# ELEMENT STIFFNESS MATRIX

- Elements are only in contact with each other at nodes.
- We imagine a node as a material particle moving during the deformation process caused by external loads affecting the structure (forces, temperatures, etc.).
- We can describe the movement of a node by giving the components of the displacement vectors.

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# ELEMENT STIFFNESS MATRIX

| Type of structure | Number of degrees of freedom | Displacements | | | Rotations | | |
|---|---|---|---|---|---|---|---|
| | $N_D$ | $u_x$ | $u_y$ | $u_z$ | $\varphi_x$ | $\varphi_y$ | $\varphi_z$ |
| plane truss | 2 | • | • | | | | |
| space truss | 3 | • | • | • | | | |
| plane frame | 3 | • | • | | | | • |
| space frame | 6 | • | • | • | • | • | • |
| grillwork | 3 | | | • | • | • | |
| 2D element | 2 | • | • | | | | |
| plate | 3 | | | • | • | • | |
| shell | 6 | • | • | • | • | • | • |
| solid (brick) | 3 | • | • | • | | | |

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# ELEMENT STIFFNESS MATRIX

- All necessary components of a nodal displacement create the system of parameters called degrees of freedom. We will mark the number of degrees of freedom as $N_D$.
- In the table there is information on the number of degrees of freedom for nodes of typical engineering structures. Degrees of freedom are given as components of displacement vectors in the Cartesian coordinate system.

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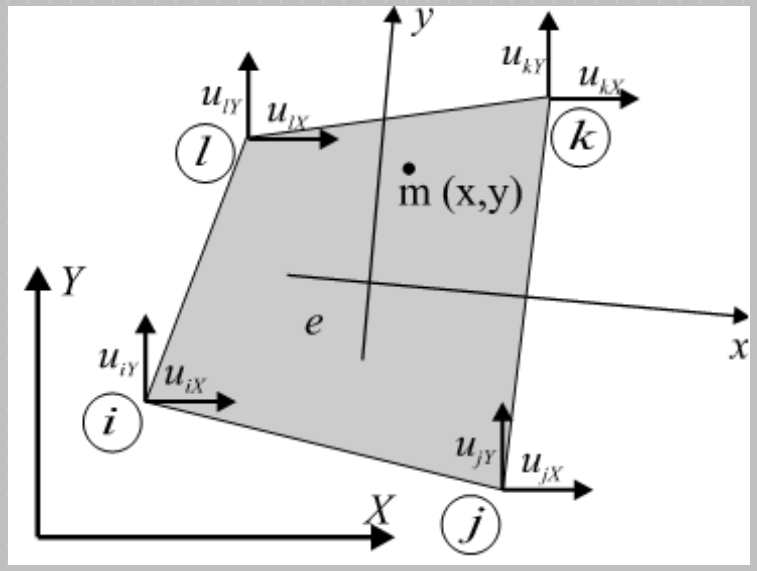
# ELEMENT STIFFNESS MATRIX




- The nodes of this element are locally numbered: $i$, $j$, $k$, $l$ and they have their global numbers resectively: $n_i$, $n_j$, $n_k$, $n_l$.

- Nodal coordinates are always given in the global coordinate system $XY$ , but for convenience we use any local coordinate system while forming an element stiffness matrix.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# ELEMENT STIFFNESS MATRIX

- We group nodal displacements in the displacement vector:

$$\mathbf{u}_i = \begin{bmatrix} u_{iX} \\ u_{iY} \end{bmatrix} \mathbf{u}_j = \begin{bmatrix} u_{jX} \\ u_{jY} \end{bmatrix} \mathbf{u}_k = \begin{bmatrix} u_{kX} \\ u_{kY} \end{bmatrix} \mathbf{u}_l = \begin{bmatrix} u_{lX} \\ u_{lY} \end{bmatrix}$$

- The set of all nodal displacements of an element forms the vector of nodal displacements of an element:

$$\mathbf{u}^e = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \\ \mathbf{u}_k \\ \mathbf{u}_l \end{bmatrix} = \begin{bmatrix} u_{iX} \\ u_{iY} \\ u_{jX} \\ u_{jY} \\ u_{kX} \\ u_{kY} \\ u_{lX} \\ u_{lY} \end{bmatrix}$$

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# ELEMENT STIFFNESS MATRIX

- The displacement of a certain point *m* within the element is written in the form of the vector:

$$\mathbf{u}(X,Y)=\begin{bmatrix} u_X(X,Y) \\ u_Y(X,Y) \end{bmatrix}$$

- If the components of vectors are defined in a local coordinate system, then we will denote them as the prime sign (′), for instance:

$$\mathbf{u}'(x,y)=\begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix}$$

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# ELEMENT STIFFNESS MATRIX

- Now we assume that the displacement of some point *m* depends on nodal displacements of an element:

$$\mathbf{u}(x,y) = \mathbf{N}^e(x,y)\mathbf{u}^e$$

- $\mathbf{N}(x,y)$ is the matrix component which depends on the coordinates of a point. The dimensions depend on element type.

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# ELEMENT STIFFNESS MATRIX

- The number of rows of the matrix $\mathbf{N}(x,y)$ is equal to the number of degrees of freedom of the point *m* and the number of columns, represents the number of degrees of freedom of the element.

$$\mathbf{u}(x,y) = \mathbf{N}^e(x,y)\mathbf{u}^e$$

can be written as:

$$\mathbf{u}(x,y) = \begin{bmatrix} \mathbf{N}_i(x,y) & \mathbf{N}_j(x,y) & \mathbf{N}_k(x,y) & \mathbf{N}_l(x,y) \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \\ \mathbf{u}_k \\ \mathbf{u}_l \end{bmatrix}$$

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# ELEMENT STIFFNESS MATRIX

- $\mathbf{N}_i(x,y)$ ... $\mathbf{N}_l(x,y)$ (matrices of shape functions) are quadratic matrices containing functions which show the influence of the displacements of nodes $i$ ... $l$ on the displacement of the point $m$.
- These are *shape functions* or *displacement functions* and they are very important for the formulation of FEM equations.

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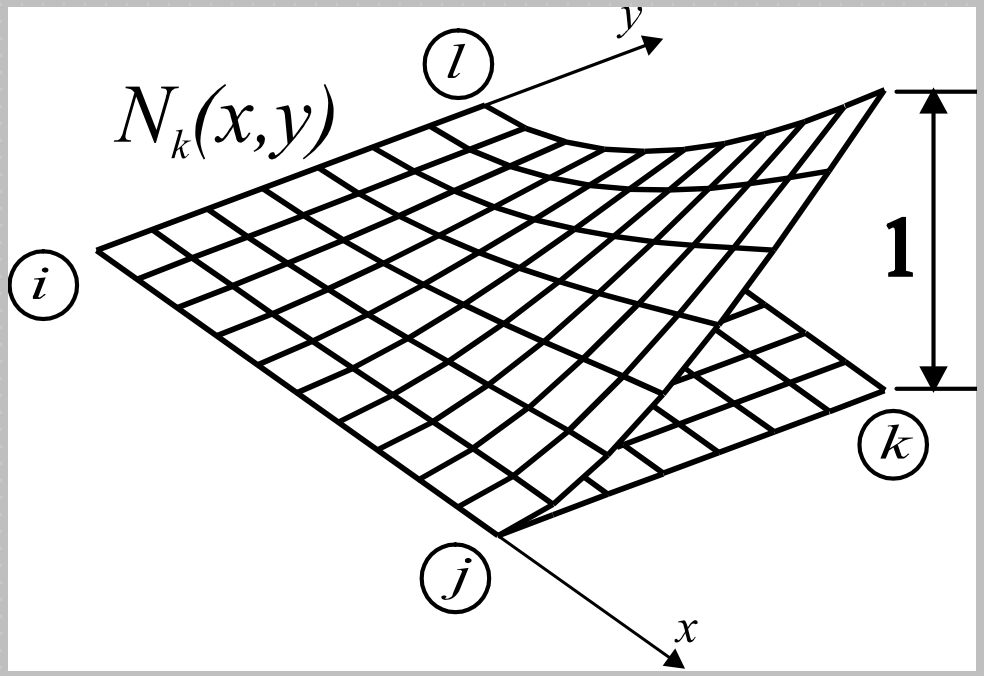
# ELEMENT STIFFNESS MATRIX




- The deformation of the element surface whose the $k$ node is displaced by a unit in the direction perpendicular to this element.

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# ELEMENT STIFFNESS MATRIX

- If we imagine that the point $m$ is at a node, then its displacements should be equal to the displacements of this node, but the displacements of other nodes should not have any influence on them:

$$N_p(x_q, y_q) = \delta_{pq}$$

where $p$ and $q$ represent any local number of nodes and $\delta_{pq}$ is a „Kronecker delta”:

$$\delta_{pq} = \begin{cases} 1 & \text{- when } p=q, \\ 0 & \text{- when } p \neq q \end{cases}$$

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# ELEMENT STIFFNESS MATRIX

- Conditions of type $N_p(x_q, y_q) = \delta_{pq}$ allow us to determine the coefficients of shape functions.
- We will consider some other conditions which have to be fulfilled by functions $N_p(x,y)$ in later parts of this chapter.

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# ELEMENT STIFFNESS MATRIX

- Substituting $\mathbf{u}(x,y)=\mathbf{N}^e(x,y)\mathbf{u}^e$ for $\boldsymbol{\sigma} = \mathbf{D}\cdot\boldsymbol{\varepsilon}$ we calculate the components of the element strain vector:

$$\boldsymbol{\varepsilon}=\mathscr{D}\cdot\mathbf{N}^e\left(x,y\right)\mathbf{u}^e$$

- $\mathscr{D}$ is the matrix with dimensions $3\times N_D$ for both plane stress and plane strain or $6\times N_D$ for three-dimensional problems containing differential operators coming from the definition of strain euqation.

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# ELEMENT STIFFNESS MATRIX

- For a two-dimensional problem, $N_D$=2 and the matrix of differential operators has the following form:

$$\mathcal{D} = \begin{bmatrix} \partial_x & 0 \\ 0 & \partial_y \\ \partial_y & \partial_x \end{bmatrix}$$

where $\partial_x$ signifies differentiation

with respect to $x$: $\partial_x = \dfrac{\partial}{\partial x}$

and $\partial_y$ with respect to $y$.

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# ELEMENT STIFFNESS MATRIX

We assume the notations:

$$\mathscr{D} \cdot \mathbf{N}^e(x, y) = \mathbf{B}^e(x, y)$$

and consistently

$$\mathscr{D} \cdot \mathbf{N}_i(x, y) = \mathbf{B}_i(x, y) \quad .. \quad \mathscr{D} \cdot \mathbf{N}_l(x, y) = \mathbf{B}_l(x, y)$$

They simplify further transformations.

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# ELEMENT STIFFNESS MATRIX

- After taking into consideration these notations, relation $\boldsymbol{\varepsilon} = \mathscr{D} \cdot \mathbf{N}^e(x, y)\mathbf{u}^e$

  can be presented as: $\boldsymbol{\varepsilon} = \mathbf{B}^e(x, y)\mathbf{u}^e$

- The matrix $\mathbf{B}(x,y)$ has dimensions $3 \times n^e{}_D$ or $6 \times n^e{}_D$ for three-dimensional problems of stress.

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# ELEMENT STIFFNESS MATRIX

- For a quadrilateral element in a 2D problem, matrix $\mathbf{B}(x,y)$ has dimensions 3×8. As with matrix $\mathbf{N}(x,y)$, we now similarly divide the matrix $\mathbf{B}(x,y)$ into blocks:

$$\mathbf{B}^e(x,y) = \begin{bmatrix} \mathbf{B}_i(x,y) & \mathbf{B}_j(x,y) & \mathbf{B}_k(x,y) & \mathbf{B}_l(x,y) \end{bmatrix}$$

- Matrices $\mathbf{B}_i$ ... $\mathbf{B}_l$ are matrices containing strain shape functions of nodes $i$ ... $l$,
- $\mathbf{B}^e(x,y)$ is the matrix containing strain shape functions of the element e.

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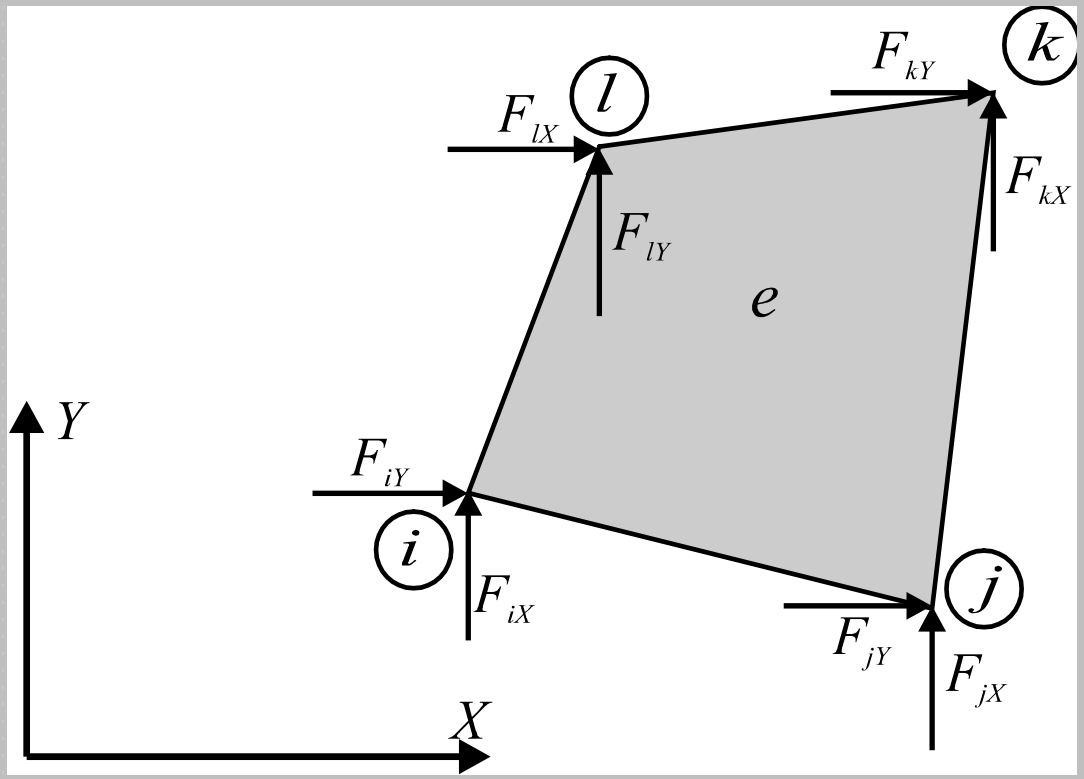
# NODAL FORCES AND DISPLACEMENTS




- The arrangement of nodal forces over the element.

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# NODAL FORCES AND DISPLACEMENTS

- Now we collect the components of nodal forces into the nodal force vector:

$$\mathbf{f}_i = \begin{bmatrix} F_{iX} \\ F_{iY} \end{bmatrix} \quad \mathbf{f}_j = \begin{bmatrix} F_{jX} \\ F_{jY} \end{bmatrix}$$

$$\mathbf{f}_k = \begin{bmatrix} F_{kX} \\ F_{kY} \end{bmatrix} \quad \mathbf{f}_l = \begin{bmatrix} F_{lX} \\ F_{lY} \end{bmatrix}$$

$$\mathbf{f}^e = \begin{bmatrix} \mathbf{f}_i \\ \mathbf{f}_j \\ \mathbf{f}_k \\ \mathbf{f}_l \end{bmatrix} = \begin{bmatrix} F_{iX} \\ F_{iY} \\ F_{jX} \\ F_{jY} \\ F_{kX} \\ F_{kY} \\ F_{lX} \\ F_{lY} \end{bmatrix}$$

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# NODAL FORCES AND DISPLACEMENTS

- We apply the principle of virtual work treating the nodal forces as the external loads on an element.
- The element is loaded both on its inside and boundary and we denote the load which depends on the coordinates of a point as follows:

$$\mathbf{q}(x,y)=\begin{bmatrix} q_x(x,y) \\ q_y(x,y) \end{bmatrix}$$

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# NODAL FORCES AND DISPLACEMENTS

- We divide constitutive Equation $\boldsymbol{\sigma} = \mathbf{D} \cdot \boldsymbol{\varepsilon}$ into parts in order to consider initial strains and stresses:

$$\boldsymbol{\sigma} = \mathbf{D}\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0\right) + \boldsymbol{\sigma}_0$$

- $\boldsymbol{\varepsilon}_0$ is the initial strain vector (for example, caused by temperature loads)
- $\boldsymbol{\sigma}_0$ is the initial stress vector (eg. residual stresses).

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# NODAL FORCES AND DISPLACEMENTS

- Now we re-write equation $\sum_{i=1}^{n} \mathbf{P}_i \cdot \bar{\mathbf{u}}_i = E_\sigma$ expressing the equality of external and internal work for the element in equilibrium:

$$\left(\mathbf{u}^e\right)^{\mathsf{T}} \mathbf{f}^e + \int_{\mathrm{A}} \mathbf{u}(x,y)^{\mathsf{T}} \mathbf{q}(x,y) d\mathscr{A} = \int_{\mathscr{V}} \boldsymbol{\varepsilon}^{\mathsf{T}} \boldsymbol{\sigma} \, d\mathscr{V}$$

- The left side of this equation represents external work, the right side denotes internal work for this element.
- $\mathscr{A}$ represents the surface of an element and $\mathscr{V}$ is its volume.

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# NODAL FORCES AND DISPLACEMENTS

Using equations:

$$\left(\mathbf{u}^e\right)^{\mathrm{T}}\mathbf{f}^e+\int_{\mathscr{A}}\mathbf{u}(x,y)^{\mathrm{T}}\mathbf{q}(x,y)d\mathscr{A}=\int_{\mathscr{V}}\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\sigma}\,d\mathscr{V}$$

$$\boldsymbol{\sigma}=\mathbf{D}\left(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_{\mathbf{o}}\right)+\boldsymbol{\sigma}_{\mathbf{o}}\qquad\boldsymbol{\varepsilon}=\mathbf{B}^e(x,y)\mathbf{u}^e$$

$$\mathbf{u}(x,y)=\begin{bmatrix}\mathbf{N}_i(x,y) & \mathbf{N}_j(x,y) & \mathbf{N}_k(x,y) & \mathbf{N}_l(x,y)\end{bmatrix}\begin{bmatrix}\mathbf{u}_i\\ \mathbf{u}_j\\ \mathbf{u}_k\\ \mathbf{u}_l\end{bmatrix}$$

we get:

$$\left(\mathbf{u}^e\right)^{\mathrm{T}}\mathbf{f}^e+\int_{\mathscr{A}}\left(\mathbf{N}^e\mathbf{u}^e\right)^{\mathrm{T}}\mathbf{q}d\mathscr{A}=\int_{\mathscr{V}}\left(\mathbf{B}^e\mathbf{u}^e\right)^{\mathrm{T}}\left[\mathbf{D}\left(\mathbf{B}^e\mathbf{u}^e-\boldsymbol{\varepsilon}_{\mathbf{o}}\right)+\boldsymbol{\sigma}_{\mathbf{o}}\right]d\mathscr{V}$$

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# NODAL FORCES AND DISPLACEMENTS

- After the transformation we obtain its final form:

$$\mathbf{f}^e = \mathbf{K}^e\mathbf{u}^e - \mathbf{f}_q^e - \mathbf{f}_{\varepsilon_o}^e + \mathbf{f}_{\sigma_o}^e$$

$$\mathbf{f}_q^e = \int_{\mathcal{A}} \left(\mathbf{N}^e\right)^{\mathrm{T}} \mathbf{q}\, d\mathcal{A}$$ - nodal forces (external loads)

$$\mathbf{f}_{\varepsilon_o}^e = \int_{\mathcal{V}} \left(\mathbf{B}^e\right)^{\mathrm{T}} \mathbf{D}\boldsymbol{\varepsilon}_\mathbf{0}\, d\mathcal{V}$$ - nodal forces (initial strain)

$$\mathbf{f}_{\sigma_o}^e = \int_{\mathcal{V}} \left(\mathbf{B}^e\right)^{\mathrm{T}} \boldsymbol{\sigma}_\mathbf{0}\, d\mathcal{V}$$ - nodal forces (initial stress)

$$\mathbf{K}^e = \int_{\mathcal{V}} \left(\mathbf{B}^e\right)^{\mathrm{T}} \mathbf{D}\,\mathbf{B}^e\, d\mathcal{V}$$ - element stiffness matrix

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# ELEMENT STIFFNESS MATRIX

- The matrix $\mathbf{K}^e$ can be divided into a block of quadratic matrices $\mathbf{K}^e{}_{qp}$ describing the influence of the displacement of the node $q$ on the forces at the node $p$:

$$\mathbf{K}^e_{pq} = \int_{\mathcal{V}} \left(\mathbf{B}^e_p\right)^{\mathsf{T}} \mathbf{D}\, \mathbf{B}^e_q d\,\mathcal{V}$$

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# ELEMENT STIFFNESS MATRIX

- There are 4×4=16 blocks in the stiffness matrix of the element with four nodes. Since the stiffness matrix is symmetrical, it means that $\mathbf{K}^e = \left(\mathbf{K}^e\right)^{\mathrm{T}}$ which comes from equation $\mathbf{K}^e = \int_{\mathcal{V}} \left(\mathbf{B}^e\right)^{\mathrm{T}} \mathbf{D}\mathbf{B}^e d\mathcal{V}$ and it is a simple

consequence of the Betti reciprocal theorem of work; then blocks $\mathbf{K}^e{}_{qp}$ have to realise the conditions:

$$\mathbf{K}^e_{qp} = \left(\mathbf{K}^e_{pq}\right)^{\mathrm{T}}$$

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# ELEMENT STIFFNESS MATRIX

- Equation $\mathbf{K}^e = \int_{\mathcal{V}} \left(\mathbf{B}^e\right)^{\mathrm{T}} \mathbf{D}\mathbf{B}^e d\mathcal{V}$ or $\mathbf{K}^e_{pq} = \int_{\mathcal{V}} \left(\mathbf{B}^e_p\right)^{\mathrm{T}} \mathbf{D}\mathbf{B}^e_q d\mathcal{V}$ represents a key step in formulating equilibrium equations of the structure but the stiffness matrix has not always been determined this way. For simple elements such as a truss element or a frame element, some other ways (sometimes simpler) of obtaining relation $\mathbf{f}^e = \mathbf{K}^e\mathbf{u}^e - \mathbf{f}^e_q - \mathbf{f}^e_{\varepsilon_o} + \mathbf{f}^e_{\sigma_o}$ exist.

  We will show these in next chapters.

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# ELEMENT STIFFNESS MATRIX

- If all transformations leading to

$$\mathbf{K}^e = \int_{\mathcal{V}} \left(\mathbf{B}^e\right)^{\mathrm{T}} \mathbf{D}\, \mathbf{B}^e \, d\mathcal{V}$$

have been done in the local coordinate system ($xyz$), then the resulting stiffness matrix should be transformed to the global coordinate system ($XYZ$).

This transformation is achieved by multiplying the matrix $\mathbf{K}^{e\prime}$ (prime sign denotes a matrix in the local coordinate system) by the transformation matrix of the element.

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$$\mathbf{K}^e = \mathbf{R}^e \mathbf{K}'^e \left(\mathbf{R}^e\right)^{\mathrm{T}}$$

$$\mathbf{R}^e = \begin{bmatrix} \mathbf{R}_i & & & \\ & \mathbf{R}_j & & \\ & & \mathbf{R}_k & \\ & & & \ddots \end{bmatrix}$$

- $\mathbf{R}_i$ ... $\mathbf{R}_k$ - transformation matrices of nodes $i$ ... $k$.
- The transformation matrices of the nodes contain cosines of angles between the axes of the global and local coordinate systems.

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# ELEMENT STIFFNESS MATRIX

$$\mathbf{R}_i = \begin{bmatrix} C_{xX} & C_{xY} & C_{xZ} \\ C_{yX} & C_{yY} & C_{yZ} \\ C_{zX} & C_{zY} & C_{zZ} \end{bmatrix} \qquad C_{xY} = \cos(\alpha_{xY}) \quad \text{etc.}$$

- $\alpha_{xY}$ is the angle between the $x$ axis of the local coordinate system and the $Y$ axis of the global system.

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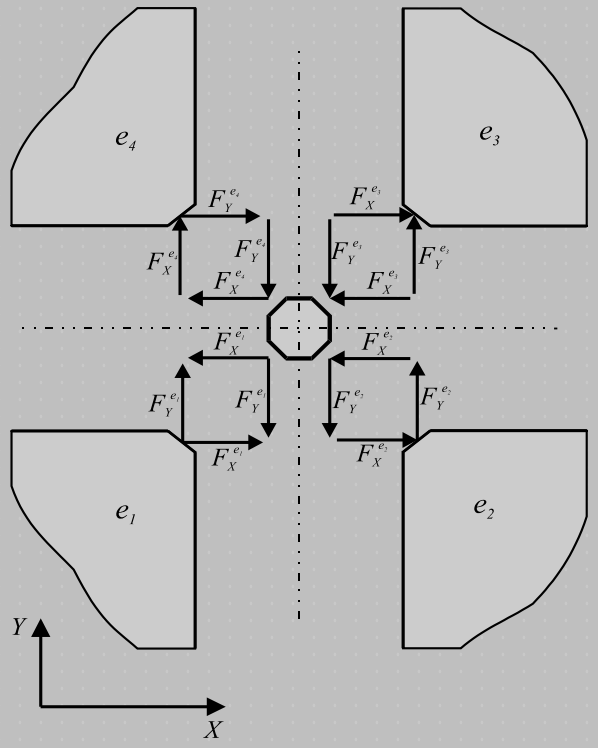
# AGGREGATION

- Relation $\mathbf{f}^e = \mathbf{K}^e\mathbf{u}^e - \mathbf{f}_q^e - \mathbf{f}_{\varepsilon_o}^e + \mathbf{f}_{\sigma_o}^e$ allows us to write equilibrium equations of a node in the form containing nodal displacements as unknown.
- Let us imagine a node as an independent part of a construction and disconnect elements from nodes in order to show nodal forces.

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# AGGREGATION

- We write a set of equilibrium equations of the node in the scalar form:

$$\sum_{k=1}^{E_n} F_X^{e_k} = 0 \quad \sum_{k=1}^{E_n} F_Y^{e_k} = 0 \quad \sum_{k=1}^{E_n} F_Z^{e_k} = 0$$

- For nodes with rotational degrees of freedom, the equations of moments will be necessary:

$$\sum_{k=1}^{E_n} M_X^{e_k} = 0 \quad \sum_{k=1}^{E_n} M_Y^{e_k} = 0 \quad \sum_{k=1}^{E_n} M_Z^{e_k} = 0$$

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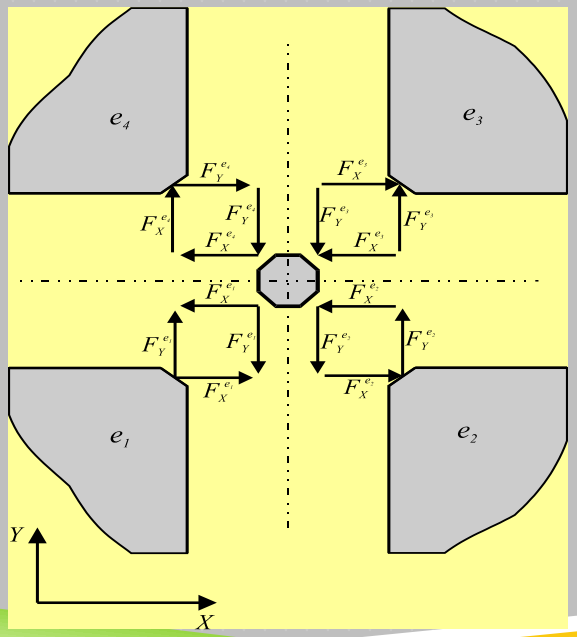
# AGGREGATION

- In previous equation summation is required for all elements connected to the node, hence indices $e_1$, $e_2$ ... are numbers of elements connected to the node, $E_n$ is the number of elements connected to the node $n$.
- We insert relationship $\mathbf{f}^e = \mathbf{K}^e\mathbf{u}^e - \mathbf{f}_q^e - \mathbf{f}_{\varepsilon_o}^e + \mathbf{f}_{\sigma_o}^e$ into equilibrium equations of forces and moments remembering to change the sign of the nodal forces coming from the change of sense of the forces acting on the element and node:

$$-\sum_{k=1}^{E_n} \mathbf{f}_n^{e_k} = 0$$

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# AGGREGATION

- In equation $-\sum_{k=1}^{E_n} \mathbf{f}_n^{e_k} = 0$ , symbol $\mathbf{f}_n^{e_k}$ defines only these components of vector which act on the node $n$.
- We convert this equation into a more convenient form:

$$\sum_{k=1}^{E_n} \mathbf{K}_n^{e_k} \mathbf{u}_n^{e_k} = \mathbf{p}_n^{e_k} \quad , \qquad \mathbf{p}^e = \mathbf{f}_q^e + \mathbf{f}_{\varepsilon_o}^e - \mathbf{f}_{\sigma_o}^e$$

is the vector of the nodal forces due to external loads, initial strains and stresses.

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# AGGREGATION

$$\sum_{k=1}^{E_n} F_X^{e_k} = 0 \qquad \sum_{k=1}^{E_n} F_Y^{e_k} = 0 \qquad \sum_{k=1}^{E_n} F_Z^{e_k} = 0 \qquad \sum_{k=1}^{E_n} M_X^{e_k} = 0 \qquad \sum_{k=1}^{E_n} M_Y^{e_k} = 0 \qquad \sum_{k=1}^{E_n} M_Z^{e_k} = 0$$

- Arranging equations for every node of the structure similar to above eqution, we obtain a set of equations which allow to calculate nodal displacements. Since summation is done for the elements in above equations (the force vectors which belong to this node), formation of a set of equations based on the equilibrium of successive nodes is not effective.

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# AGGREGATION

- Ordering nodes and degrees of freedom is necessary for this operation. So far we have used local numbers for nodes of elements $i$, $j$, $k$, $l$ .., but introducing global numeration of nodes is necessary while building the global set of equations.
- Let $n_i$ stand for a global number of the node represented by the local number $i$ and let $s_p$ be a global number of degrees of freedom $p$.

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# AGGREGATION

- Now we form a rectangular matrix of connections of the element $e - \mathbf{A}^e$.
- The number of rows of the matrix is equal to the global number of degrees of freedom of the structure $N_k$, the number of columns is equal to the number of degrees of freedom of the element $e - N_D^e$ .

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# AGGREGATION

- Most components of the matrix $\mathbf{A}^e$ are equal to zero apart from the components having the value of 1 which are situated in rows $s_p$ and columns $p$. Hence, the structure of the matrix contains information about connections between the element and nodes or being more exact about the relationship between the degree of freedom of the element and the global degree of freedom of the structure.

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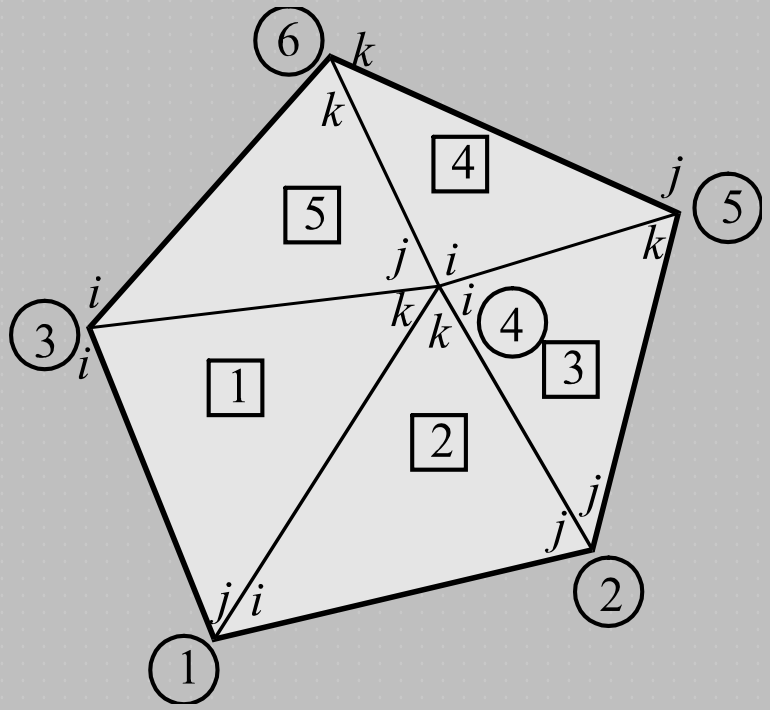
# AGGREGATION

- The formation of the connection matrix can be most easily studied on the following example.




This is a plate divided into five triangular elements. The plate has six nodes numbered from 1 to 6, every element has a local notation of nodes $i$, $j$, $k$.

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# AGGREGATION

- The table shows global numeration of degrees of freedom of a 2D element of the plate.

| Node number | Global numbers of degrees of freedom of nodes | |
|---|---|---|
| $n$ | $u_{nX}$ | $u_{nY}$ |
| 1 | 1 | 2 |
| 2 | 3 | 4 |
| 3 | 5 | 6 |
| 4 | 7 | 8 |
| 5 | 9 | 10 |
| 6 | 11 | 12 |

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# AGGREGATION

- The table shows the dependence between local and global degrees of freedom:

| Element number | Global numbers of degrees of freedom of element $S_p$ - allocation vector | | | | | |
|---|---|---|---|---|---|---|
| $e$ | $u_{Xi}$ | $u_{Yi}$ | $u_{Xj}$ | $u_{Yj}$ | $u_{Xk}$ | $u_{Yk}$ |
| | *1* | *2* | *3* | *4* | *5* | *6* |
| 1 | 5 | 6 | 1 | 2 | 7 | 8 |
| 2 | 1 | 2 | 3 | 4 | 7 | 8 |
| 3 | 7 | 8 | 3 | 4 | 9 | 10 |
| 4 | 7 | 8 | 9 | 10 | 11 | 12 |
| 5 | 5 | 6 | 7 | 8 | 11 | 12 |

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# AGGREGATION

- Hence the connection matrix created for element No 3 will have the following form:

all zero elements are neglected for clarity

$$\mathbf{A}^3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} & & & & & \\ & & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & & \\ & & & & & \\ 1 & & & & & \\ & 1 & & & & \\ & & & & 1 & \\ & & & & & 1 \\ & & & & & \\ & & & & & \end{bmatrix} \end{array}$$

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# AGGREGATION

- Multiplying the nodal force vector of an element by the connection matrix causes the transfer of suitable blocks of the local vector to the global vector. Now simple addition of these vectors is possible:

$$\sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{f}^e = \sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{K}^e \mathbf{u}^e = \sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{p}^e$$

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# AGGREGATION

- Here it is necessary to express the nodal displacement vector of elements by means of the global vector:

$$\mathbf{u}^e = \left(\mathbf{A}^e\right)^{\mathrm{T}} \mathbf{u}$$

which should be put into

$$\sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{f}^e = \sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{K}^e \mathbf{u}^e = \sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{p}^e$$

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# AGGREGATION

- Finally, we obtain the system of equations in the form:

$$\sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{K}^e \left(\mathbf{A}^e\right)^{\mathrm{T}} \mathbf{u} = \sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{p}^e$$

or in a shorter form $\mathbf{K}\mathbf{u} = \mathbf{p}$

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# AGGREGATION

- Matrix $\mathbf{K} = \sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{K}^e \left(\mathbf{A}^e\right)^{\mathsf{T}}$ is called the global stiffness matrix of a structure, vector $\mathbf{p} = \sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{p}^e$ is the global vector of nodal forces of the structure, the vector $\mathbf{u}$ containing the displacement of all nodes is the global displacement vector.
- A similar method of aggregation is described in the book written by Rakowski and Kacprzyk (1993) where matrix $\mathbf{A}^{\mathsf{T}}$ is called the adhesion matrix.

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# AGGREGATION

- The method of aggregation using the adhesion matrix is not suitable for computer implementation because it uses the big matrix $\mathbf{A}^e$. It is more effective to exploit information which is contained in allocation vectors. Vectors for the previous example are included in previous table. The aggregation method using allocation vectors will be presented in the next chapter.

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# REMARKS REGARDING THE SHAPE FUNCTIONS

- Functions approximating the displacement field within elements which are in fact shape function cannot be chosen in freely.
- They should fulfil some conditions which decide about the quality of these functions or their usefulness for approximation of displacements, strains and stresses.
- We quote these criteria after Zienkiewicz (1972).

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# REMARKS REGARDING THE SHAPE FUNCTIONS

### *A. Criteria of rigid body movements*

- The displacement function chosen should be in such a way that it should not permit straining of an element to occur when the nodal displacements are caused by a rigid body displacement.

### *B. Criterion of strain stability*

- The shape function should enable the constant field of strains in an element to appear.

### *C. Criterion of strain agreement*

- The displacement functions should be so chosen that the strains at the interface between elements are finite.

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# REMARKS REGARDING THE SHAPE FUNCTIONS

- Criteria (A) and (B) seem to be obvious. Since some components of strain (or stress) can be zero, then approximation functions should be able to reproduce these problems. Constant and linear parts of polynomials which we often use to build a shape structure, assure realisation of conditions (A) and (B). Criterion (B) is the generalisation of criterion (A).
- Criterion (C) requires that shape functions should assure continuity of derivatives to the degree which is lower by one than differential operators being in the matrix $\mathscr{D}$:

$$\boldsymbol{\varepsilon} = \mathscr{D} \cdot \mathbf{N}^{e}\left(x, y\right)\mathbf{u}^{e}$$

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# REMARKS REGARDING THE SHAPE FUNCTIONS

We explain this using the following example.

- In the 2D problem of a plate, the strains are defined by the first derivation of the displacement function

(comp. $\boldsymbol{\varepsilon} = \mathscr{D} \cdot \mathbf{N}^e(x, y)\mathbf{u}^e$ ),

because the displacement field has to be continuous on the boundary between elements and displacements functions have to be of class $C^0$.

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# REMARKS REGARDING THE SHAPE FUNCTIONS

- For plate elements, the curvatures given by the second order derivatives take the role of displacements .
- Hence the displacement function of a plate should assure continuity both of the surfaces of a plate deflection and its first derivations inside and on the boundaries between elements.
- Then the displacement field should be continuous and smooth within the plate (class $C^1$).

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- Criteria (A) and (B) have to be realised, criterion (C) does not. For instance, the shape function of plate elements does not often achieve the condition of continuity (continuity of the first derivations on boundaries of elements).
- If all criteria are realised, then we say that the described elements are ‘adjust ones’
- If only criteria (A) and (B) are achieved, then elements are called ‘not adjust ones’.

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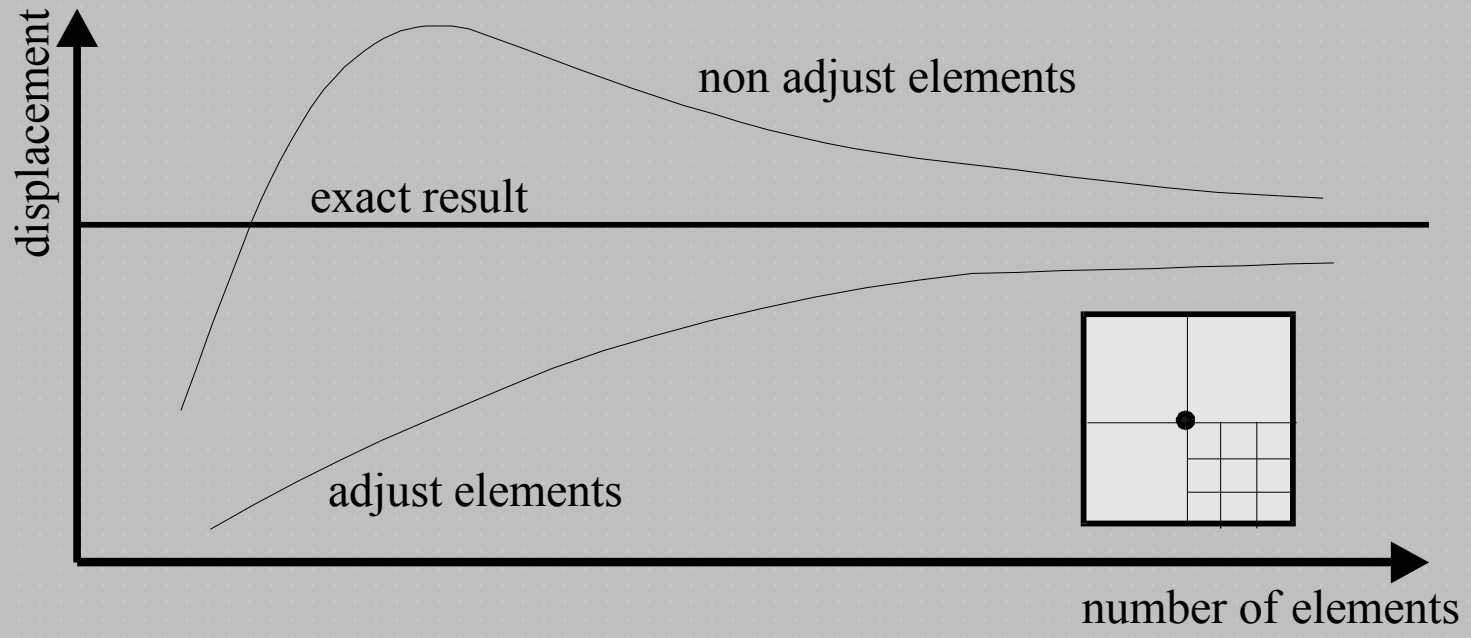


- The precision of calculations for incompatible and non-incompatible elements depending on the number of elements.

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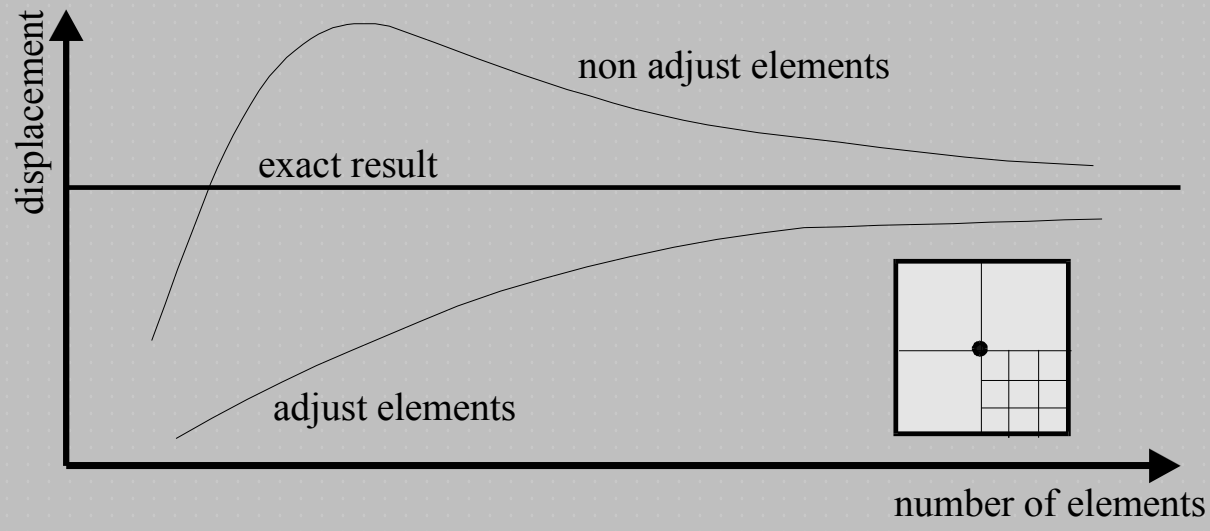


The result of applying 'adjust' and 'not adjust' elements to discretization of a structure is presented in above figure. The convergence of results obtained with the help of the different types of elements which are used for discretization of a quadratic plate is shown in the same figure.

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- Apart from the three listed criteria we can also add some others which determine the choice of approximation polynomials. This choice should assure isotropy with respect to axes of a coordinate system. We will show this using the example of building shape functions of plate elements (two- and three-dimensional problems).

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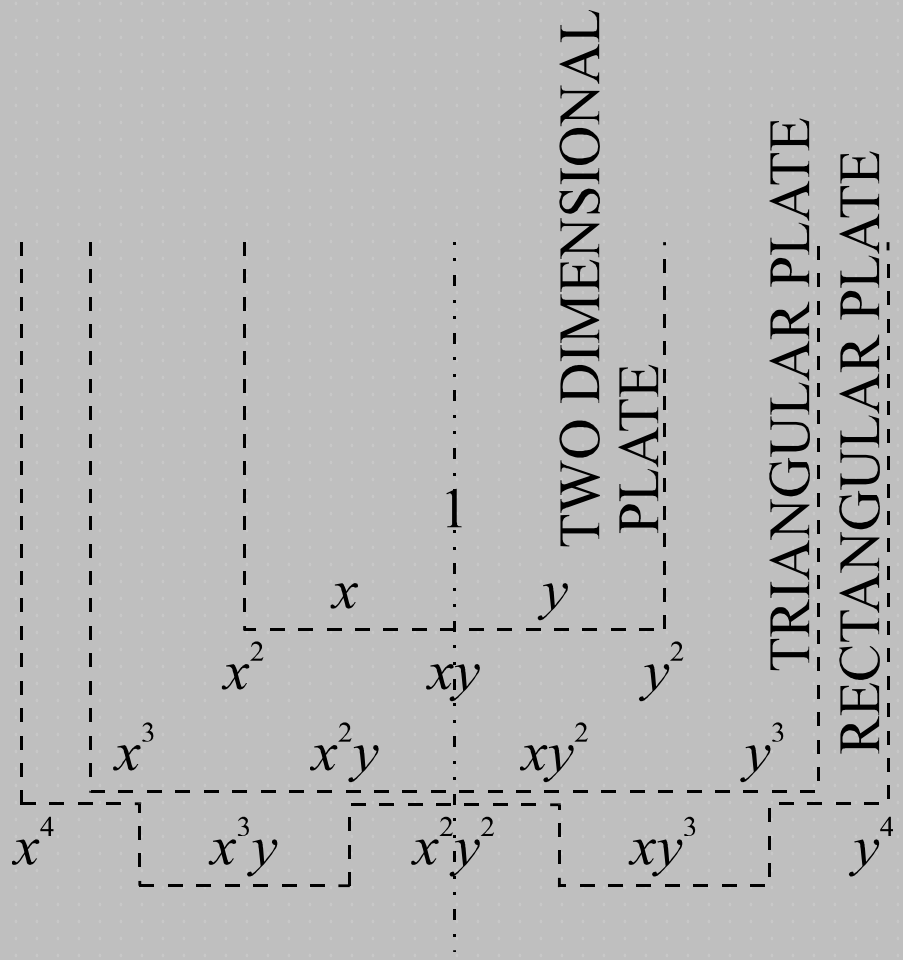
- If we present approximation polynomials in the form of Pascal's triangle, then the choice of part of this triangle should be symmetrical in with respect to its axes.

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- We can use Hermitte (will be described further) and Lagrange polynomials (Zienkiewicz (1972)), but we always have to maintain the condition of isotropy.
- There is a long list of references as far as shape functions are concerned but we recommend the following books: Bathe (1996), Rakowski and Kacprzyk (1993), Rao (1982), Zienkiewicz and Taylor (1994).

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