## Advanced topics in Finite Element Method

 3D truss structuresJerzy Podgórski

## Introduction

Although 3D truss structures have been around for a long time, they have been used very rarely until now. They are difficult to solve. Though a series method simplifying the calculation of internal forces has been devised for statically determined plane trusses, in case of space trusses, only the method of nodal equilibrium has remained.

## Introduction

Large sets of equations which are generated by this method for space trusses have discouraged engineers from designing this type of structure. 3D structures looking like trusses, in fact, are seldom trusses. For instance support columns of overhead power lines are most often space frames because they keep their geometric stability with bent elements which don't exist in classical trusses.

## Introduction

Both the use of computers and new methods of statics analysis of a structure making use of new technical possibilities (the finite element method is one of the main methods among them) have enabled considerable progress in designing space trusses.
One of the most popular uses of these structures is in structural roofs. Examples of space trusses are presented in the next slide.

## Introduction

the space structure


## Introduction

the tower


## Notation and basic relations

The node of a space truss has three degrees of freedom because in order to describe its movement, we have to give three components of a displacement vector. The displacement vector and forces acting on an element of the space truss are shown in next figure. As in previous presentation components of forces and displacements vector are collected in column matrices which will be called vectors.

Notation and basic relations

- nodal displacements vector of the first node $i$ in the global and local coordinate system:

$$
\mathbf{u}_{i}=\left[\begin{array}{l}
u_{i_{X}} \\
u_{i z} \\
u_{i z}
\end{array}\right] \quad \quad \mathbf{u}_{i}==\left[\begin{array}{l}
u_{i z} \\
u_{i z} \\
u_{i z}
\end{array}\right]
$$

- vector of nodal forces acting at the first node $i$ in the global and local coordinate system:

$$
\mathbf{f}_{i}=\left[\begin{array}{c}
F_{i X} \\
F_{i Y} \\
F_{i Z}
\end{array}\right] \quad \mathbf{f}_{i}^{\prime}=\left[\begin{array}{c}
F_{i x} \\
F_{i y} \\
F_{i z}
\end{array}\right]
$$

## Notation and basic relations

The above vectors form forces and displacements vectors of an element:

- vector of the nodal displacements of an element $e$ with the node $i$ and $j$ is written in the global and local coordinate system:

$$
\mathbf{u}^{e}=\left[\begin{array}{c}
\mathbf{u}_{i} \\
\mathbf{u}_{j}
\end{array}\right]=\left[\begin{array}{c}
u_{i X} \\
u_{i Y} \\
u_{i Z} \\
u_{j X} \\
u_{j Y} \\
u_{j Z}
\end{array}\right] \quad \quad \mathbf{u}^{\prime e}=\left[\begin{array}{l}
\mathbf{u}_{i}^{\prime} \\
\mathbf{u}_{j}{ }_{j}
\end{array}\right]=\left[\begin{array}{l}
u_{i x} \\
u_{i y} \\
u_{i z} \\
u_{j x} \\
u_{j y} \\
u_{j z}
\end{array}\right]
$$

## Notation and basic relations

- vector of the nodal forces of an element $e$ in the global and local system:

$$
\mathbf{f}^{e}=\left[\begin{array}{c}
\mathbf{f}_{i} \\
\mathbf{f}_{j}
\end{array}\right]=\left[\begin{array}{c}
F_{i X} \\
F_{i Y} \\
F_{i Z} \\
F_{j X} \\
F_{j Y} \\
F_{j Z}
\end{array}\right] \quad \mathbf{f}^{\prime e}=\left[\begin{array}{c}
\mathbf{f}^{\prime}{ }_{i} \\
\mathbf{f}^{\prime}{ }_{j}
\end{array}\right]=\left[\begin{array}{c}
F_{i x} \\
F_{i y} \\
F_{i z} \\
F_{j x} \\
F_{j y} \\
F_{j z}
\end{array}\right]
$$

## Notation and basic relations

Global system

## Notation and basic relations



# The element stiffness matrix of a space truss 

The relationship between nodal forces and nodal displacements for a space truss is identical to that for a plane truss if we analyse it in the local coordinate system. Obviously, the third force is $F_{i z}$ or $F_{j z}$ but the equilibrium equation of moments with respect to the $y$ axis results in the zero value of this force:

## The element stiffness matrix of a space truss

$\sum F_{x}=F_{i x}+F_{j x}=0 \rightarrow F_{i x}=-F_{j x}$
$\sum F_{y}=F_{i j}+F_{j y}=0 \xrightarrow{\text { after considering eq. } \mathrm{f}} F_{i y}=0$
$\sum F_{z}=F_{i z}+F_{j z}=0 \xrightarrow{\text { after considering eq. } \mathrm{e}} F_{i z}=0$
$\sum M_{x}=0$
$\sum M_{y}=-F_{j z} L=0 \rightarrow F_{j z}=0$
$\sum M_{z}=-F_{j y} L=0 \rightarrow F_{j y}=0$

## The element stiffness matrix of a space truss

The relationship between an axial force and displacements which is identical to the relation presented in previous presentation (comp. $N=\frac{E A}{L}\left(u_{j_{j}}-u_{x_{x}}\right)$ ) allows us to express the searched dependence as follows:

$$
\begin{gathered}
\mathbf{f}^{\prime e}=\mathbf{K}^{\prime e} \mathbf{u}^{\prime e} \\
\mathbf{K}^{\prime e}=\left[\begin{array}{rr}
\mathbf{J}^{\prime} & -\mathbf{J}^{\prime} \\
-\mathbf{J}^{\prime} & \mathbf{J}^{\prime}
\end{array}\right] \quad \mathbf{J}^{\prime}=\frac{E A}{L}\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right]
\end{gathered}
$$

## The element stiffness matrix of a space truss

The transformation from the local system to the global one will be done analogously to the transformation performed in case of a 2D truss (Eqn. $\left.\mathbf{f}_{i}^{\prime}=\left(\mathbf{R}_{i}\right)^{\top} \mathbf{f}_{i}, \mathbf{K}^{e}=\mathbf{R}^{e} \mathbf{K}^{e c}\left(\mathbf{R}^{e}\right)^{\top}, \mathbf{f}^{e}=\mathbf{K}^{e} \mathbf{u}^{e}\right)$.
In order to complete the transformation to the global system, we need the rotation matrix of a node $\mathbf{R}_{i}$, and then we can determine components of $\mathbf{J}$ similar to $\mathbf{J}=\frac{E A}{L}\left[\begin{array}{ll}c^{2} & s c \\ s c & s^{2}\end{array}\right]$

## The element stiffness matrix of a space truss



The truss
element arrangement with regard to the global coordinate system.

# The element stiffness matrix of a space truss 

Since the location of the $y$ and $z$ axes of the local system is not essential for truss elements, we will choose the direction of the $y$ axis in such a way that it will always be parallel to the $X Y$ plane of the global system but for bars parallel to the $Z$ axis there will be an additional assumption that the $y$ axis is parallel to the $Y$ axis.

## The element stiffness matrix of a space truss

The rotation from the local coordinate system to the global one will be composed of two intermediate rotations. First, we rotate the system $x y z$ to the intermediate system $x$ " $y$ " $z$ " selected so that the $x^{\prime \prime}$ axis is parallel to the $X Y$ plane and next we rotate the system $x$ " $y$ " $z$ " by an angle $\gamma$ so that the $x$ " and $X$ axes are parallel.

## The element stiffness matrix of a space truss

The first rotation around the $y$ axis gives the following result:

$$
\begin{array}{ll}
{\left[\begin{array}{l}
u_{x^{\prime \prime}} \\
u_{y^{\prime \prime}} \\
u_{z^{\prime \prime}}
\end{array}\right]=\left[\begin{array}{ccc}
c_{\beta} & 0 & -s_{\beta} \\
0 & 1 & 0 \\
s_{\beta} & 0 & c_{\beta}
\end{array}\right]\left[\begin{array}{l}
u_{x} \\
u_{y} \\
u_{z}
\end{array}\right] \quad \text { or }} & \mathbf{u}^{\prime \prime}=\mathbf{R}_{\beta} \mathbf{u}^{\prime} \\
c_{\beta}=\cos \beta=\frac{L^{\prime \prime}}{L} & L_{X}=X_{j}-X_{i} \\
L_{Y}=Y_{j}-Y_{i} & L^{\prime \prime}=\sqrt{L_{X}^{2}+L_{Y}^{2}} \\
s_{\beta}=\sin \beta=\frac{L_{Z}}{L} & L_{Z}=Z_{j}-Z_{i}
\end{array}
$$

## The element stiffness matrix of a space truss

The second rotation around the $z$ axis leads the equations to the global system:

$$
\begin{gathered}
{\left[\begin{array}{l}
u_{X} \\
u_{Y} \\
u_{Z}
\end{array}\right]=\left[\begin{array}{ccc}
c_{\gamma} & -s_{\gamma} & 0 \\
s_{\gamma} & c_{\gamma} & 0 \\
0 & 0 & 1
\end{array}\right]\left[\begin{array}{l}
u_{x^{\prime \prime}} \\
u_{y^{\prime \prime}} \\
u_{z^{\prime \prime}}
\end{array}\right] \quad \text { or } \quad \mathbf{u}=\mathbf{R}_{\gamma} \mathbf{u}^{\prime \prime}} \\
c_{\gamma}=\cos \gamma=\frac{L_{X}}{L^{\prime \prime}} \quad s_{\gamma}=\sin \gamma=\frac{L_{Y}}{L^{\prime \prime}}
\end{gathered}
$$

when L' $=0$ we assume $\gamma=0$, hence $c_{\gamma}=1$ and $s_{\gamma}=0$

## The element stiffness matrix of a space truss

The composition of both rotations which means putting $\mathbf{k}^{1 e}=\left[\begin{array}{cc}\mathbf{J}^{\prime} & -\mathbf{J}^{\prime} \\ -\mathbf{J}^{\prime} & \mathbf{J}^{\prime}\end{array}\right]$ into $\mathbf{u}^{\prime \prime}=\mathbf{R}_{\beta} \mathbf{u}^{\prime}$, gives the searched rotation matrix of a node

$$
\mathbf{u}_{i}=\mathbf{R}_{i \gamma} \mathbf{R}_{i \beta} \mathbf{u}_{i}^{\prime}
$$

where $\quad \mathbf{R}_{i}=\mathbf{R}_{i \gamma} \mathbf{R}_{i \beta}$

## The element stiffness matrix of a space truss

After multiplying matrices, we obtain the final form of the rotation matrix $\mathbf{R}_{i}$ :

$$
\mathbf{R}_{i}=\left[\begin{array}{ccc}
c_{\gamma} c_{\beta} & -s_{\gamma} & -c_{\gamma} s_{\beta} \\
s_{\gamma} c_{\beta} & c_{\gamma} & -s_{\gamma} s_{\beta} \\
s_{\beta} & 0 & 0
\end{array}\right]
$$

We calculate the transformation of the block J of the element stiffness matrix of the space truss from the local coordinate system to the global one as in previous presentation

## The element stiffness matrix of a space truss

$\mathbf{J}=\mathbf{R}_{i} \mathbf{J}^{\prime}\left(\mathbf{R}_{i}\right)^{\mathrm{T}}$
Inserting relations $\boldsymbol{J}^{\prime}=\frac{E A}{L}\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right]$ and $\mathbf{u}_{i}=\mathbf{R}_{i \gamma} \mathbf{R}_{i \beta} \mathbf{u}_{i}^{\prime}$ into the above equation we obtain:

$$
\mathbf{J}=\frac{E A}{L}\left[\begin{array}{ccc}
\left(c_{\gamma} c_{\beta}\right)^{2} & c_{\gamma} s_{\gamma}\left(c_{\beta}\right)^{2} & c_{\gamma} c_{\beta} s_{\beta} \\
c_{\gamma} s_{\gamma}\left(c_{\beta}\right)^{2} & \left(s_{\gamma} c_{\beta}\right)^{2} & s_{\gamma} c_{\beta} c_{\beta} \\
c_{\gamma} c_{\gamma} s_{\beta} & s_{\gamma} c_{\beta} s_{\beta} & \left(s_{\beta}\right)^{2}
\end{array}\right]
$$

## The element stiffness matrix of a space truss

After the introduction of a convenient notation:

$$
C_{X}=\frac{L_{X}}{L} \quad C_{Y}=\frac{L_{Y}}{L} \quad C_{Z}=\frac{L_{Z}}{L}
$$

which are called direction cosines of an element, we obtain a very simple form of the block J of the stiffness matrix:

$$
\mathbf{J}=\frac{E A}{L}\left[\begin{array}{ccc}
C_{X}^{2} & C_{X} C_{Y} & C_{X} C_{Z} \\
C_{X} C_{Y} & C_{Y}^{2} & C_{Y} C_{Z} \\
C_{X} C_{Z} & C_{Y} C_{Z} & C_{Z}^{2}
\end{array}\right] \quad \mathbf{K}^{e}=\left[\begin{array}{rr}
\mathbf{J} & -\mathbf{J} \\
-\mathbf{J} & \mathbf{J}
\end{array}\right]
$$

# The temperature loads 

## for 3D truss

Forming a loads vector of a truss for concentrated forces is identical to forming it for a 2D truss. We will also not discuss the vector $\mathbf{p}$. We will discuss the vector of nodal forces due to a temperature load.

$$
\mathbf{f}^{\prime e t}=E A \alpha_{t} \Delta t_{o}\left[\begin{array}{c}
1 \\
0 \\
0 \\
-1 \\
0 \\
0
\end{array}\right]
$$

This vector in the local coordinate system is similar to the components for a plane truss.

The temperature loads

## for 3D truss

The transformation to the global system proceeds in agreement with $\mathbf{f}^{e}=\mathbf{R}^{e} \mathbf{f}^{1 e}$ in the following way: $\mathbf{f}^{e t}=\mathbf{R}^{e} \mathbf{f}^{\text {1et }}$
$\mathbf{R}^{e}=\left[\begin{array}{cc}\mathbf{R}_{i} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{j}\end{array}\right]$
Since a truss element is straight, $\mathbf{R}_{i}=\mathbf{R}_{j}$, where the matrix $\mathbf{R}_{i}$ is defined by $\mathbf{R}_{i}=\mathbf{R}_{i \gamma} \mathbf{R}_{i \beta}$.

The temperature loads

## for 3D truss

After inserting $\mathbf{R}_{i}=\mathbf{R}_{i \gamma} \mathbf{R}_{i \beta}$ into $\mathbf{f}^{\prime t t}$ and multiplying them, we obtain:


$$
\mathbf{f}^{e t}=E A \alpha_{t} \Delta t_{o}\left[\begin{array}{c}
C_{X} \\
C_{Y} \\
C_{Z} \\
-C_{X} \\
-C_{Y} \\
-C_{Z}
\end{array}\right]
$$

The remaining procedure is identical to the one employed in case of a plane truss.

The boundary element

In previous presentation, we explained widely different types of boundary conditions and also elastic boundary elements. Since they are very useful elements for modelling many different boundary conditions, we will pay more attention to them in this chapter concentrating on differences between plane and space trusses.

The boundary element

We will discuss the most general elastic element with stiffness $k_{b}$ dropping with respect to axes of the global system with the angles $\alpha_{x}, \alpha_{y}, \alpha_{z}$.

$$
c_{X}=\cos \alpha_{X} \quad c_{Y}=\cos \alpha_{Y} \quad c_{Z}=\cos \alpha_{Z}
$$

The stiffness matrix of this element in the local system is analogous to the matrix stiffness of an ordinary truss element but this element has three degrees of freedom, so the stiffness matrix contains only one block J’.

## The boundary element

$\mathbf{K}^{\prime b}=k_{b}\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right]$
Transforming this element to the global coordinate system we obtain a matrix which is similar to the one obtained for a plane truss:

$$
\mathbf{K}^{b}=k_{b}\left[\begin{array}{ccc}
c_{X}^{2} & c_{X} c_{Y} & c_{X} c_{Z} \\
c_{X} c_{Y} & c_{Y}^{2} & c_{Y} c_{Z} \\
c_{X} c_{Z} & c_{Y} c_{Z} & c_{Z}^{2}
\end{array}\right]
$$

## The boundary element

Boundary elements can form for example, an element with three different types of stiffness $k_{x}, k_{y}, k_{z}$ parallel to axes of the local system

$$
\text { xyz: } \mathbf{K}^{\prime b}=\left[\begin{array}{ccc}
k_{x} & 0 & 0 \\
0 & k_{y} & 0 \\
0 & 0 & k_{z}
\end{array}\right]
$$

The transformation of this matrix to the global system is analogous to the transformation of the block J': $\mathbf{J}=\mathbf{R}_{i} \mathbf{J}^{\prime}\left(\mathbf{R}_{i}\right)^{\mathbf{T}}$

## Stresses and Internal forces

We present here equations to calculate stresses and internal forces in an element:
$\sigma_{x}=E\left(\varepsilon-\varepsilon_{t}\right)=\frac{E}{L}\left[\left(u_{j x}-u_{i x}\right)-L\left(\alpha_{t} \Delta t_{o}\right)\right]$
or $\quad \sigma_{x}=E\left(\varepsilon-\varepsilon_{t}\right)=\frac{E}{L}\left[\left(u_{j x}-u_{i x}\right)-L\left(\alpha_{t} \Delta t_{o}\right)\right]$
The transformation of the vector to the global system gives the relationship:

$$
\sigma_{x}=\frac{E}{L}\left[\begin{array}{llllll}
-1 & 0 & 0 & 1 & 0 & 0
\end{array}\right]\left(\mathbf{R}^{e}\right)^{\mathrm{T}} \mathbf{u}^{e}-E \alpha_{t} \Delta t_{o}
$$

## Stresses and Internal forces

$$
\sigma_{x}=\frac{E}{L}\left[\begin{array}{llllll}
-1 & 0 & 0 & 1 & 0 & 0
\end{array}\right]\left(\mathbf{R}^{e}\right)^{\mathrm{T}} \mathbf{u}^{e}-E \alpha_{t} \Delta t_{o}
$$

After multiplication it gives components of direct stress in an element as follows:

$$
\sigma_{x}=E\left\{\left[\begin{array}{ll}
-\mathbf{c}^{\mathrm{T}} & \mathbf{c}^{\mathrm{T}}
\end{array}\right]\left(\mathbf{R}^{e}\right)^{\mathrm{T}} \mathbf{u}^{e} \frac{1}{L}-\alpha_{t} \Delta t_{o}\right\}
$$

where $\mathbf{c}$ is the vector of element direction cosines: $\quad \mathbf{c}^{\mathrm{T}}=\left[c_{X} c_{Y} c_{Z}\right]$

## Stresses and Internal forces

Calculating the normal force consists of integrating stresses on the surface of a cross section with an assumption of homogeneity of the stress field

$$
N=\sigma_{x} A=E A\left\{\frac{1}{L}\left[-\mathbf{c}^{\mathrm{T}} \quad \mathbf{c}^{\mathrm{T}}\right]\left(\mathbf{R}^{e}\right)^{\mathrm{T}} \mathbf{u}^{e}-\alpha_{t} \Delta t_{o}\right\}
$$

The remaining support reactions are calculated with the help of $\mathbf{r}=\mathbf{K u}-\mathbf{p}$. We can do it exactly in the same way as it has been done for the 2 D truss.

