Finite Element Method
2D frame systems

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Introduction

The correct choice of model for a structure is very important for quality and exactness of the results obtained. The choice of frame or truss (for example, a truss with fixed nodes) is often subjective and it depends on experience and intuition of the analyst.

In this chapter, we will present the following model of a bar structure - a 2D frame which gives more possibilities.

Introduction

The element of a 2D frame is more general than a truss element presented in Chapter 2 because with help of this element we can also model ideal truss structures (articulated connection of elements at nodes). We can simply say that a frame is a structure whose bars can be bent while truss elements can be only compressed and stretched.

Introduction

It has the following consequences:

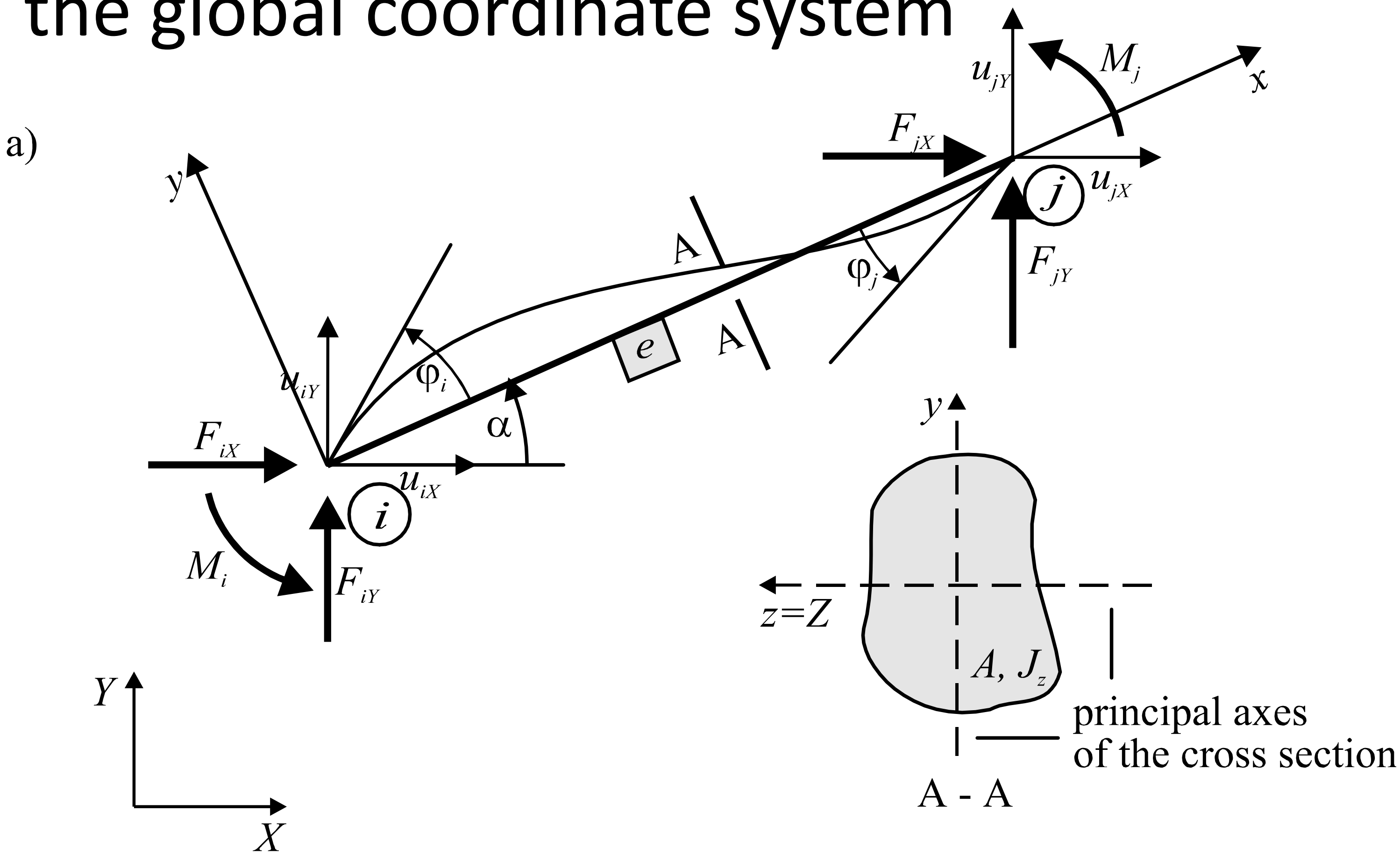
- bar (an element) of a frame can be loaded between nodes,
- modelling of different types of loads is possible,
- connection of an element with a node can be a fixed or an articulated connection,
- node of a 2D frame has 3 degrees of freedom,

Introduction

In the case of plane frames, we will neglect index *Z* of rotation angles in our notation because all rotation angles on the plane *XY* (which we will use to describe the structure) are rotations with respect to the *Z* axis. Let us assume that a frame element is straight and homogeneous which means that it is made from a homogeneous material and has a constant cross section.

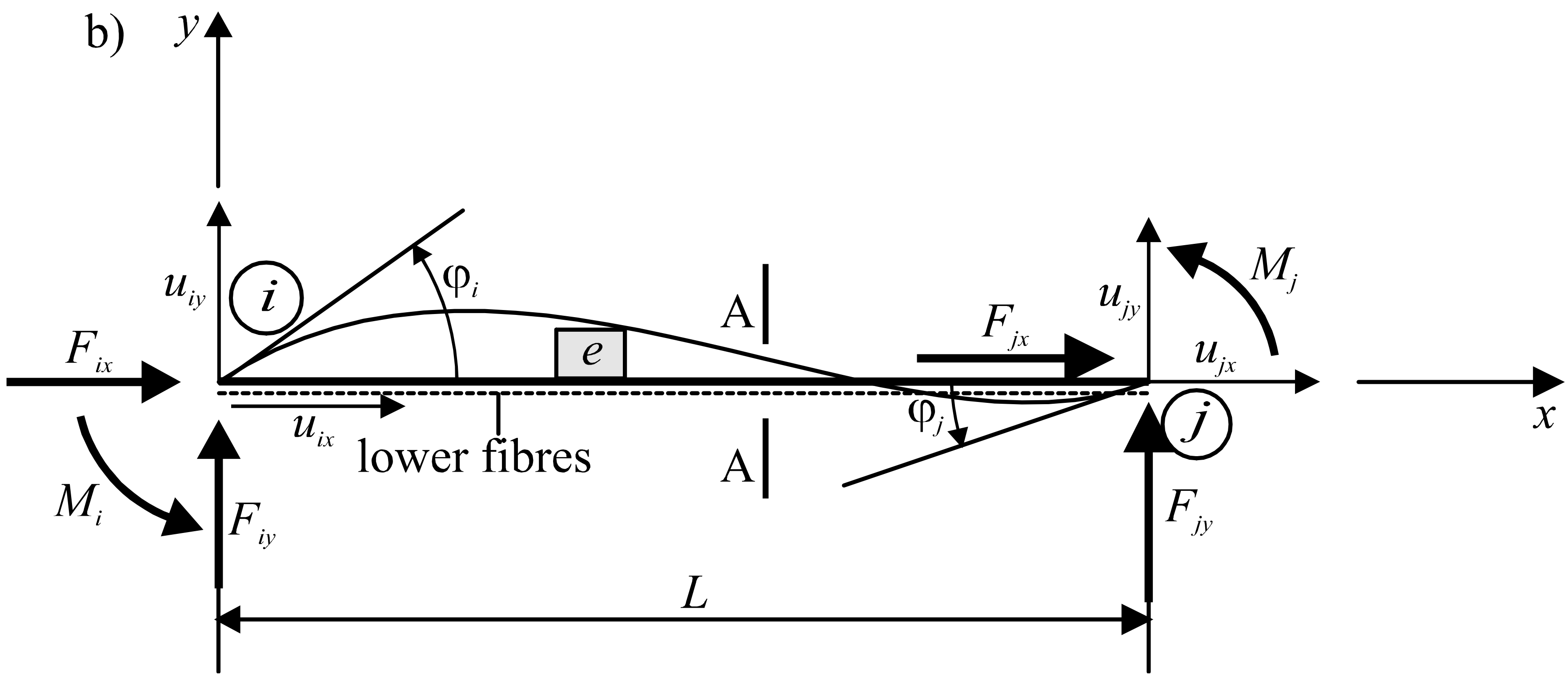
Introduction

the global coordinate system



Introduction

the local coordinate system



The element stiffness matrix for a 2D frame

We group nodal displacements and forces in column matrices just as we did previously. They are called vectors:

- displacement and nodal forces vector of the first node *i* and the last node *j* in the local system

$$\mathbf{u}'_i = \begin{bmatrix} u_{ix} \\ u_{iy} \\ \varphi_i \end{bmatrix} \qquad \mathbf{u}'_j = \begin{bmatrix} u_{jx} \\ u_{jy} \\ \varphi_j \end{bmatrix} \qquad \mathbf{f}'_i = \begin{bmatrix} F_{ix} \\ F_{iy} \\ M_i \end{bmatrix} \qquad \mathbf{f}'_j = \begin{bmatrix} F_{jx} \\ F_{jy} \\ M_j \end{bmatrix}$$

The element stiffness matrix for a 2D frame

- element displacement vector in the local coordinate system

$$\mathbf{u}'^{e} = \begin{bmatrix} \mathbf{u}'_i \\ \mathbf{u}'_j \end{bmatrix} = \begin{bmatrix} u_{ix} \\ u_{iy} \\ \varphi_i \\ u_{jx} \\ u_{jy} \\ \varphi_j \end{bmatrix}$$

element forces vector in the local coordinate system

$$\mathbf{f}'^{e} = \begin{bmatrix} \mathbf{f}'_i \\ \mathbf{f}'_j \end{bmatrix} = \begin{bmatrix} F_{ix} \\ F_{iy} \\ M_i \\ F_{jx} \\ F_{jy} \\ M_j \end{bmatrix}$$

The element stiffness matrix for a 2D frame

We can also describe all the vectors formulated above in the global system:

$$\mathbf{u}_i = \begin{bmatrix} u_{iX} \\ u_{iY} \\ \varphi_i \end{bmatrix} \qquad \mathbf{u}_j = \begin{bmatrix} u_{jX} \\ u_{jY} \\ \varphi_j \end{bmatrix} \qquad \mathbf{f}_i = \begin{bmatrix} F_{iX} \\ F_{iY} \\ M_i \end{bmatrix} \qquad \mathbf{f}_j = \begin{bmatrix} F_{jX} \\ F_{jY} \\ M_j \end{bmatrix}$$

$$\mathbf{u}^e = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \end{bmatrix} = \begin{bmatrix} u_{iX} \\ u_{iY} \\ \varphi_i \\ u_{jX} \\ u_{jY} \\ \varphi_j \end{bmatrix} \qquad \mathbf{f}^e = \begin{bmatrix} \mathbf{f}_i \\ \mathbf{f}_j \end{bmatrix} = \begin{bmatrix} F_{iX} \\ F_{iY} \\ M_i \\ F_{jX} \\ F_{jY} \\ M_j \end{bmatrix}$$

The element stiffness matrix for a 2D frame

As in the previous chapters, the relationship between nodal forces and nodal displacements will be of great importance. This relation (analogous to a truss) in the local coordinate system has the form:

$$\mathbf{K}'^{e}\,\mathbf{u}'^{e} = \mathbf{f}'^{e}$$

and in the global system

$$\mathbf{K}^{e}\,\mathbf{u}^{e} = \mathbf{f}^{e}$$

The element stiffness matrix for a 2D frame

At the moment, we will concentrate on searching for the stiffness matrix **K'**^{e} in the local coordinate system and next its transformation to the global system.

Equilibrium equations of the element lead to the following relations between nodal forces:

$$\sum F_x = F_{ix} + F_{jx} = 0 \quad \rightarrow \quad F_{ix} = -F_{jx}$$

$$\sum F_y = F_{iy} + F_{jy} = 0 \qquad \sum M_i = M_i + M_j + F_{jy}\, L = 0$$

The element stiffness matrix for a 2D frame

It has been shown that three equations are unable to calculate six components of the vector . The discussion concerning element strains will provide these missing equations. The deformation caused by the axial forces F_{ix} and F_{jx} is identical to the deformation of a truss element, hence we take advantage of previously determined dependence:

$$N = \frac{EA}{L}\left(u_{jx} - u_{ix}\right) \qquad F_{ix} = \frac{EA}{L}\left(u_{ix} - u_{jx}\right) \qquad F_{jx} = \frac{EA}{L}\left(-u_{ix} + u_{jx}\right)$$

The element stiffness matrix for a 2D frame

We will obtain the remaining equations when we consider the flexural deformation of an element and the relationship between shearing forces and bending moments. The well-known relationship between curvature and bending moment is:

$$\frac{1}{\rho} = \frac{\dfrac{d^2 y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} = \frac{M(x)}{EJ_z}$$

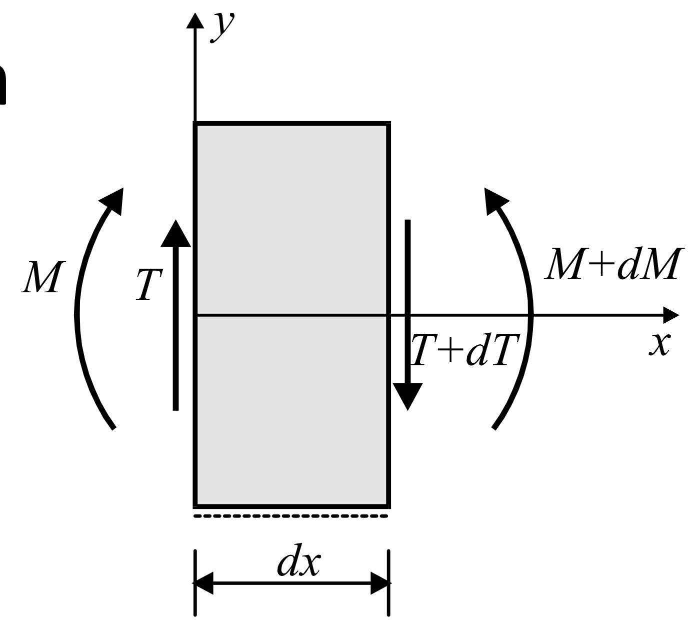
The element stiffness matrix for a 2D frame

ρ - the radius of a curvature,

E - Young's modulus of a material,

J_z - the moment of inertia of an element cross section

$$\frac{1}{\rho}=\frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}=\frac{M(x)}{EJ_z}$$



The element stiffness matrix for a 2D frame

The equilibrium of one section of a bar in bending gives the equation:

$$T(x) = \frac{dM(x)}{dx}$$

Since we are dealing with linear structures with small deflections, we assume *dy/dx*<< 1, which simplifies the relationship between curvature and bending moment to the well-known form:

$$\frac{d^2 y}{dx^2} = \frac{M(x)}{EJ_z}$$

The element stiffness matrix for a 2D frame

The opposite sign of the right side of $\frac{d^2y}{dx^2} = \frac{M(x)}{EJ_z}$ to the one that we have usually assumed, comes from the sense of the *y* axis of the local coordinate system which is orientated anticlockwise in our assumptions.

Differentiating this equation twice, we obtain the relationship:

$$\frac{d^4y}{dx^4} = \frac{q_y(x)}{EJ_z}$$

The element stiffness matrix for a 2D frame

$$\frac{d^4 y}{dx^4} = \frac{q_y(x)}{EJ_z}$$

$q_y(x)$ denotes the distributed load which is perpendicular to the axis of an element. Here the element is free from nodal loads, thus $q_y \equiv 0$

Finally, we obtain the set of differential equations:

$$\frac{d^4 y}{dx^4} = 0 \qquad \frac{d^2 y}{dx^2} = \frac{M(x)}{EJ_z} \qquad \frac{d^3 y}{dx^3} = \frac{T(x)}{EJ_z}$$

The element stiffness matrix for a 2D frame

After integrating relations $\frac{d^4 y}{dx^4} = 0$ we obtain the following equations:

- bending line of the frame element:

$$y(x) = C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4$$

- bending moment:

$$M(x) = EJ_z \left[C_1 x + C_2 \right]$$

- shearing force:

$$T(x) = EJ_z C_1$$

C_1 ... C_4 are integration constants which should be determined on the basis of boundary conditions.

The element stiffness matrix for a 2D frame

We have four boundary conditions:

- at node *i* , *x*=0:

$$y(0) = u_{iy} \qquad \left.\frac{dy}{dx}\right|_{x=0} = \varphi_i$$

- at node *j* , *x*=*L*:

$$y(L) = u_{jy} \qquad \left.\frac{dy}{dx}\right|_{x=L} = \varphi_j$$

The element stiffness matrix for a 2D frame

After inserting these conditions into $y(x) = C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4$, we obtain the following values of the integration constants:

$$C_1 = \frac{6}{L^2}\left(\varphi_i + \varphi_j - 2\frac{u_{jy} - u_{iy}}{L}\right) \qquad C_3 = \varphi_i$$

$$C_2 = -\frac{1}{L}\left(4\varphi_i + 2\varphi_j - 6\frac{u_{jy} - u_{iy}}{L}\right) \qquad C_4 = u_{iy}$$

The element stiffness matrix for a 2D frame

Hence after putting the above equations into

$M(x) = EJ_z\left[C_1 x + C_2\right]$ and $T(x) = EJ_z C_1$ we obtain:

$$M_i = -M(0) = \frac{EJ_z}{L}\left[4\varphi_i + 2\varphi_j - 6\frac{u_{jy} - u_{iy}}{L}\right]$$

$$M_j = M(L) = \frac{EJ_z}{L}\left[2\varphi_i + 4\varphi_j - 6\frac{u_{jy} - u_{iy}}{L}\right]$$

$$F_{iy} = T(0) = \frac{EJ_z}{L^2}\left[6\varphi_i + 6\varphi_j - 12\frac{u_{jy} - u_{iy}}{L}\right]$$

$$F_{jy} = -T(L) = \frac{EJ_z}{L^2}\left[-6\varphi_i - 6\varphi_j + 12\frac{u_{jy} - u_{iy}}{L}\right]$$

The element stiffness matrix for a 2D frame

Finally, tabulating equations $F_{ix} = \frac{EA}{L}\left(u_{ix} - u_{jx}\right)$, $F_{jx} = \frac{EA}{L}\left(-u_{ix} + u_{jx}\right)$ and from the previous slide in a suitable sequence we obtain the stiffness matrix:

$$
\mathbf{K}'^{e} = \begin{bmatrix}
\frac{EA}{L} & & & -\frac{EA}{L} & & \\
& 12\frac{EJ_z}{L^3} & 6\frac{EJ_z}{L^2} & & -12\frac{EJ_z}{L^3} & 6\frac{EJ_z}{L^2} \\
& 6\frac{EJ_z}{L^2} & 4\frac{EJ_z}{L} & & -6\frac{EJ_z}{L^2} & 2\frac{EJ_z}{L} \\
-\frac{EA}{L} & & & \frac{EA}{L} & & \\
& -12\frac{EJ_z}{L^3} & -6\frac{EJ_z}{L^2} & & 12\frac{EJ_z}{L^3} & -6\frac{EJ_z}{L^2} \\
& 6\frac{EJ_z}{L^2} & 2\frac{EJ_z}{L} & & -6\frac{EJ_z}{L^2} & 4\frac{EJ_z}{L}
\end{bmatrix}
$$

The element stiffness matrix for a 2D frame

The relationships described by equations

$$M_i = -M(0) = \frac{EJ_z}{L}\left[4\varphi_i + 2\varphi_j - 6\frac{u_{jy} - u_{iy}}{L}\right]$$

$$M_j = M(L) = \frac{EJ_z}{L}\left[2\varphi_i + 4\varphi_j - 6\frac{u_{jy} - u_{iy}}{L}\right]$$

$$F_{iy} = T(0) = \frac{EJ_z}{L^2}\left[6\varphi_i + 6\varphi_j - 12\frac{u_{jy} - u_{iy}}{L}\right]$$

$$F_{jy} = -T(L) = \frac{EJ_z}{L^2}\left[-6\varphi_i - 6\varphi_j + 12\frac{u_{jy} - u_{iy}}{L}\right]$$

are called transformation formulae of the displacement method in structural mechanics (in some other form).

The stiffness matrix in the local coordinate system

The transfer of the matrix to the global coordinate system is done according to rules analogous to the rules described for 2D truss element. In order to obtain the transformation matrix of an element, we need $\mathbf{R}_i$ that is, the transformation matrix from the local system to the global one for the node *i*.

The stiffness matrix in the local coordinate system

Since the third degree of freedom is a rotation with respect to the *z* axis which does not change its location because it is always perpendicular to the plane *xy*, the rotation will be the same as for a truss element:

$$u_{iX} = u_{ix}\cos\alpha - u_{iy}\sin\alpha$$

$$u_{iY} = u_{ix}\sin\alpha + u_{iy}\cos\alpha$$

$$\varphi_{iZ} = \varphi_{iz} = \varphi_i$$

or

$$\mathbf{u}_i = \mathbf{R}_i \mathbf{u}_i'$$

$$\begin{bmatrix} u_{iX} \\ u_{iY} \\ \varphi_i \end{bmatrix} = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{ix} \\ u_{iy} \\ \varphi_i \end{bmatrix}$$

The stiffness matrix in the local coordinate system

In accordance that the frame element is straight, the transformation matrix of the node *j* is identical to $\mathbf{R}_i$ which leads to the final form of the element stiffness matrix:

$$\mathbf{R}^e = \begin{bmatrix} c & -s & 0 & 0 & 0 & 0 \\ s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

After multiplying matrices $\mathbf{K}^e = \mathbf{R}^e \mathbf{K}'^e \left(\mathbf{R}^e\right)^{\mathrm{T}}$ we obtain the stiffness matrix of a frame element in the global coordinate system. Unfortunately, its form is rather complex.

The stiffness matrix in the local coordinate system

$$
\mathbf{K}^e = \frac{EJ_z}{L^2}
\begin{array}{cc}
\begin{array}{cccccc} u_{ix} & u_{iy} & \varphi_{ix} & u_{jx} & u_{jy} & \varphi_{jx} \end{array} & \\
\left[\begin{array}{cccccc}
\frac{1}{L}\left(\frac{c^2}{\lambda^2}+12s^2\right) & \frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & -6s & -\frac{1}{L}\left(\frac{c^2}{\lambda^2}+12s^2\right) & -\frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & -6s \\
\frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & \frac{1}{L}\left(\frac{s^2}{\lambda^2}+12c^2\right) & 6c & -\frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & -\frac{1}{L}\left(\frac{s^2}{\lambda^2}+12c^2\right) & 6c \\
-6s & 6c & 4L & -6s & -6c & 2L \\
-\frac{1}{L}\left(\frac{c^2}{\lambda^2}+12s^2\right) & -\frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & -6s & \frac{1}{L}\left(\frac{c^2}{\lambda^2}+12s^2\right) & \frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & 6s \\
-\frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & -\frac{1}{L}\left(\frac{s^2}{\lambda^2}+12c^2\right) & -6c & \frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & \frac{1}{L}\left(\frac{s^2}{\lambda^2}+12c^2\right) & -6c \\
-6s & 6c & 2L & 6s & -6c & 4L
\end{array}\right]
&
\begin{array}{c} F_{ix} \\ F_{iy} \\ M_i \\ F_{jx} \\ F_{jy} \\ M_j \end{array}
\end{array}
$$

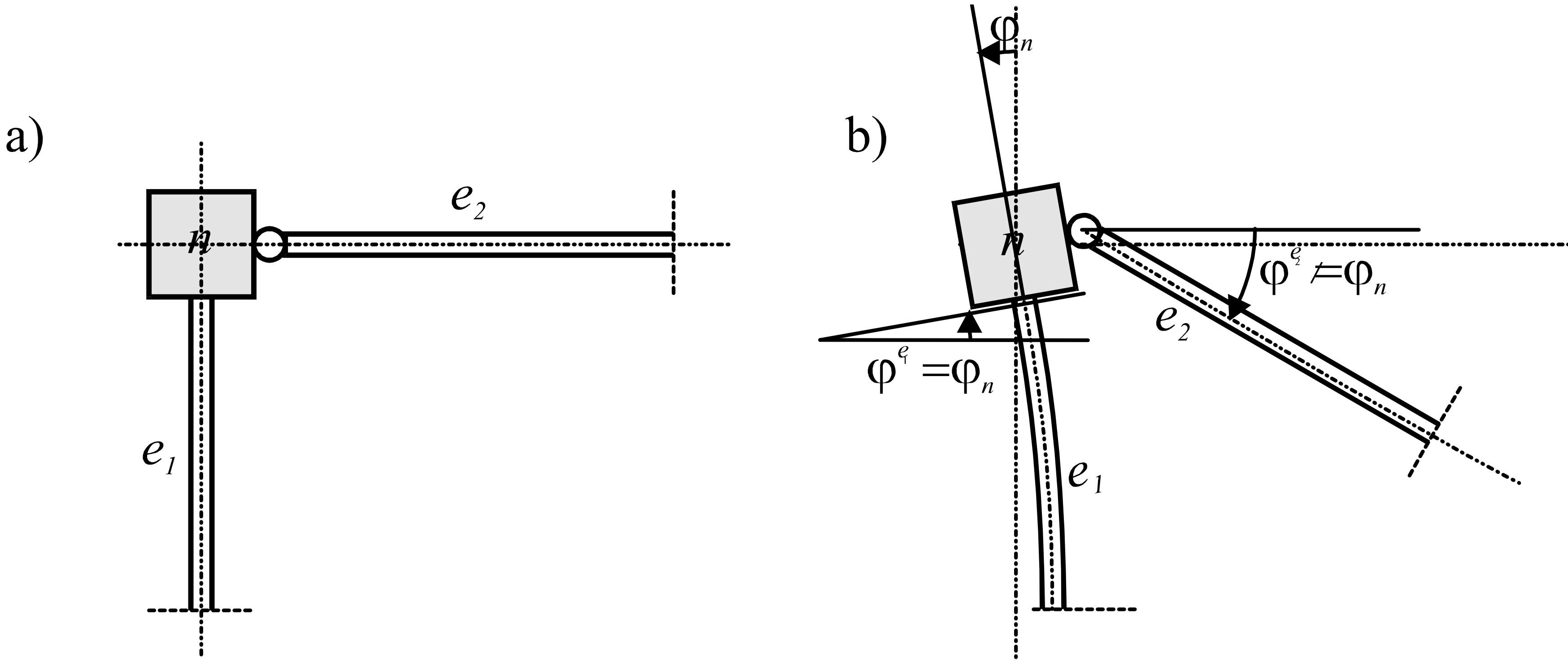
$$\lambda^2 = \frac{J_z}{AL^2} \qquad c = \cos\alpha \qquad s = \sin\alpha$$

Static reduction of the stiffness matrix

Frame elements are not always joined at a node ensuring the agreement of all displacements of nodes and in the bar section at this node. Articulated joints shown in next figure are examples of such incomplete connections.

At this joint, the angle of the nodal rotation does not influence the rotation of the element section of a node. The latter e_2 can rotate independently of the node.

Static reduction of the stiffness matrix



The element joint scheme with one element able to rotate (an articulated joint).

Static reduction of the stiffness matrix

We determine the unknown angle of the rotation of such an element using an additional equation which is given by the equilibrium condition of moments in a joint. Hence we can reduce the number of degrees of freedom of the element because the additional equilibrium condition allows us to eliminate one displacement from the set of equations.

Static reduction of the stiffness matrix

Example 1 - articulated connection.

Additional equilibrium condition of a section at the node i : $M_i = 0$ leads, after considering

equations $\mathbf{f}'_i = \begin{bmatrix} F_{ix} \\ F_{iy} \\ M_i \end{bmatrix}$, $\mathbf{f}'_j = \begin{bmatrix} F_{jx} \\ F_{jy} \\ M_j \end{bmatrix}$, $\mathbf{K}'^e \mathbf{u}'^e = \mathbf{f}'^e$

to conditions: $\frac{EJ_z}{L}\left[6\frac{u_{iy}}{L} + 4\varphi_i - 6\frac{u_{jy}}{L} + 2\varphi_j\right] = 0$

Static reduction of the stiffness matrix

$$\frac{EJ_z}{L}\left[6\frac{u_{iy}}{L}+4\varphi_i-6\frac{u_{jy}}{L}+2\varphi_j\right]=0$$

we calculate the required value of the rotation angle of the section at the node *i*:

$$\varphi_i=-\frac{3}{2}\frac{u_{iy}}{L}+\frac{3}{2}\frac{u_{jy}}{L}-\frac{1}{2}\varphi_j$$

Static reduction of the stiffness matrix

After putting this result into $\mathbf{K}'^{e}\,\mathbf{u}'^{e} = \mathbf{f}'^{e}$ and taking into consideration matrix **K’**^{e} we obtain:

$$F_{iy} = \frac{EJ_z}{L^2}\left[12\frac{u_{iy}}{L} + 6\left(-\frac{3}{2}\frac{u_{iy}}{L} + \frac{3}{2}\frac{u_{jy}}{L} - \frac{1}{2}\varphi_j\right) - 12\frac{u_{jy}}{L} + 6\varphi_j\right] = \frac{EJ_z}{L^2}\left[3\frac{u_{iy}}{L} - 3\frac{u_{jy}}{L} + 3\varphi_j\right],$$

$$F_{jy} = \frac{EJ_z}{L^2}\left[-12\frac{u_{iy}}{L} - 6\left(-\frac{3}{2}\frac{u_{iy}}{L} + \frac{3}{2}\frac{u_{jy}}{L} - \frac{1}{2}\varphi_j\right) + 12\frac{u_{jy}}{L} - 6\varphi_j\right] = \frac{EJ_z}{L^2}\left[-3\frac{u_{iy}}{L} + 3\frac{u_{jy}}{L} - 3\varphi_j\right]$$

$$M_j = \frac{EJ_z}{L}\left[6\frac{u_{iy}}{L} + 2\left(-\frac{3}{2}\frac{u_{iy}}{L} + \frac{3}{2}\frac{u_{jy}}{L} - \frac{1}{2}\varphi_j\right) - 6\frac{u_{jy}}{L} + 4\varphi_j\right] = \frac{EJ_z}{L}\left[3\frac{u_{iy}}{L} - 3\frac{u_{jy}}{L} + 3\varphi_j\right]$$

Static reduction of the stiffness matrix

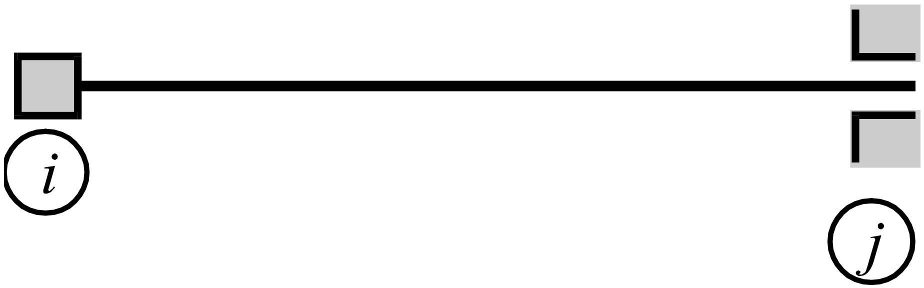
The new stiffness matrix of an element with the joint at the node *i*:

$$\mathbf{K}'^{e(3,i)} = \begin{bmatrix} \frac{EA}{L} & 0 & & -\frac{EA}{L} & 0 & 0 \\ 0 & 3\frac{EJ_z}{L^3} & & 0 & -3\frac{EJ_z}{L^3} & 3\frac{EJ_z}{L^2} \\ & & & & & \\ -\frac{EA}{L} & 0 & & \frac{EA}{L} & 0 & 0 \\ 0 & -3\frac{EJ_z}{L^3} & & 0 & 3\frac{EJ_z}{L^3} & -3\frac{EJ_z}{L^2} \\ 0 & 3\frac{EJ_z}{L^2} & & 0 & -3\frac{EJ_z}{L^2} & 3\frac{EJ_z}{L} \end{bmatrix}$$

Superscripts (3,*i*) indicate that the third degree of freedom is eliminated at the first node.

Static reduction of the stiffness matrix

Example 2 - moveable connection.



Additional equilibrium condition of a section at the node *j*: $F_{jx} = 0$ leads to condition:

$F_{ix} = 0$

it does not change the relations for the remaining nodal forces.

Static reduction of the stiffness matrix

The stiffness matrix of such an element takes the following form:

$$
\mathbf{K}'^{e(1,j)} = \begin{bmatrix}
 & & & & & \\
 & 12\frac{EJ_z}{L^3} & 6\frac{EJ_z}{L^2} & & -12\frac{EJ_z}{L^3} & 6\frac{EJ_z}{L^2} \\
 & 6\frac{EJ_z}{L^2} & 4\frac{EJ_z}{L} & & -6\frac{EJ_z}{L^2} & 2\frac{EJ_z}{L} \\
 & & & & & \\
 & -12\frac{EJ_z}{L^3} & -6\frac{EJ_z}{L^2} & & 12\frac{EJ_z}{L^3} & -6\frac{EJ_z}{L^2} \\
 & 6\frac{EJ_z}{L^2} & 2\frac{EJ_z}{L} & & -6\frac{EJ_z}{L^2} & 4\frac{EJ_z}{L}
\end{bmatrix}
$$

Superscripts (1,*j*) indicate that the first degree of freedom is eliminated at the last node.

Static reduction of the stiffness matrix

The above process is called the static reduction of a stiffness matrix. Now we will give the matrix notation of an operation leading to a reduced stiffness matrix. For the sake of simplicity, we assume that the last degree of freedom of an element is the eliminated degree of freedom.

Static reduction of the stiffness matrix

Nodal forces, nodal displacements vectors and the stiffness matrix are divided into blocks:

$$\left[\begin{array}{c} \mathbf{f}_1 \\ \hline \mathbf{f}_0 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{K}_{11} & \mathbf{K}_{10} \\ \hline \mathbf{K}_{01} & \mathbf{K}_{00} \end{array}\right] \left[\begin{array}{c} \mathbf{u}_1 \\ \hline \mathbf{u}_0 \end{array}\right]$$

where according to the symmetry of the matrix we have:

$$\mathbf{K}_{11} = \mathbf{K}_{11}^{\mathrm{T}} \qquad \mathbf{K}_{01} = \mathbf{K}_{10}^{\mathrm{T}}$$

Static reduction of the stiffness matrix

$\mathbf{K}_{00}$ is the matrix 1x1 and thus it is a scalar, the blocks and and are also scalars. The results of the multiplication of previous matrix blocks are:

$$\mathbf{f}_1 = \mathbf{K}_{11}\mathbf{u}_1 + \mathbf{K}_{10}\mathbf{u}_0 \qquad \mathbf{f}_0 = 0 = \mathbf{K}_{01}\mathbf{u}_1 + \mathbf{K}_{00}\mathbf{u}_0$$

From $\mathbf{f}_0$ equation we calculate: $\mathbf{u}_0 = -\mathbf{K}_{00}^{-1}\mathbf{K}_{01}\mathbf{u}_1$

and after inserting into $\mathbf{f}_1$ equation we obtain:

$$\mathbf{f}_1 = \mathbf{K}_{11}\mathbf{u}_1 - \mathbf{K}_{10}\mathbf{K}_{00}^{-1}\mathbf{K}_{01}\mathbf{u}_1 \quad \text{or} \quad \mathbf{f}_1 = \mathbf{K}''\mathbf{u}_1$$

K'' - the condensed element stiffness matrix.

Static reduction of the stiffness matrix

Vector of an element load still remains to be determined. We obtain it by composing both the load vector of an element with rigid connections with nodes and the vector of the load caused by displacements of nodes free from constraints:

$$\mathbf{f} = \mathbf{f}^{o} - \mathbf{f}^{u} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1^{o} \\ \mathbf{f}_0^{o} \end{bmatrix} - \begin{bmatrix} \mathbf{f}_1^{u} \\ \mathbf{f}_0^{u} \end{bmatrix}$$

Static reduction of the stiffness matrix

Since $\mathbf{f}_0 = \mathbf{f}_0^o - \mathbf{f}_0^u = 0$ then:

$$\mathbf{f}_0^u = \mathbf{f}_0^o = \mathbf{K}_{01}^o \mathbf{u}_1 + \mathbf{K}_{00}^o \mathbf{u}_0$$

and hence

$$\mathbf{u}_0 = \left(\mathbf{K}_{00}^o\right)^{-1} \mathbf{f}_0^o$$

because other displacements contained in $\mathbf{u}_1$ are equal to zero.

Static reduction of the stiffness matrix

Finally, we obtain

$$\mathbf{f}_1 = \mathbf{f}_1^o - \mathbf{K}_{10}^o \left(\mathbf{K}_{00}^o \right)^{-1} \mathbf{f}_0^o$$

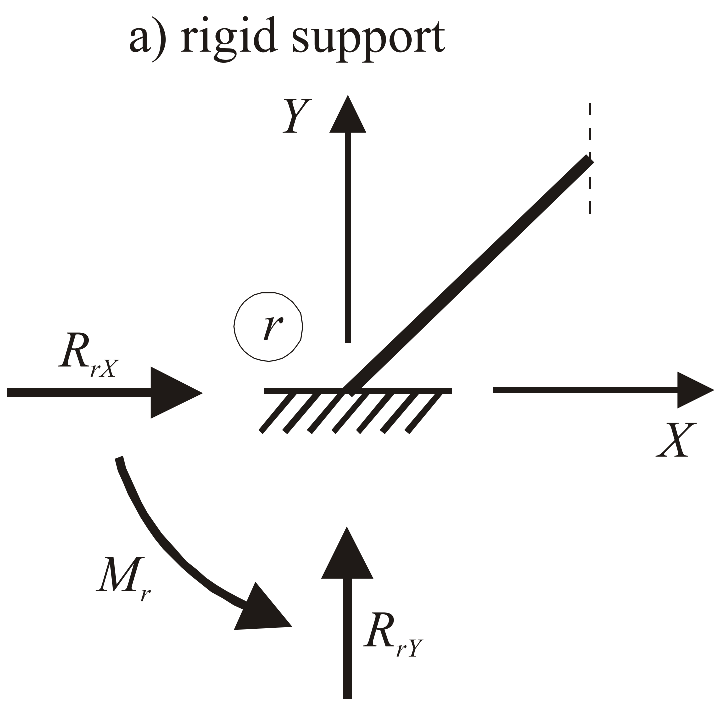
In this way, we can eliminate any degree of freedom but it requires some more complex transformations. We leave this problem to be solved by the reader.

Boundary conditions of plane frame structures

Supports for plane frames include articulated and fixed supports all listed in presentation 2. The latter ones prevent the rotation of a support node. Symbolic notation of these supports and the boundary conditions describing them are shown in next slides.

For non-typical supports, we propose to consider the use of suitable boundary elements instead of these supports.

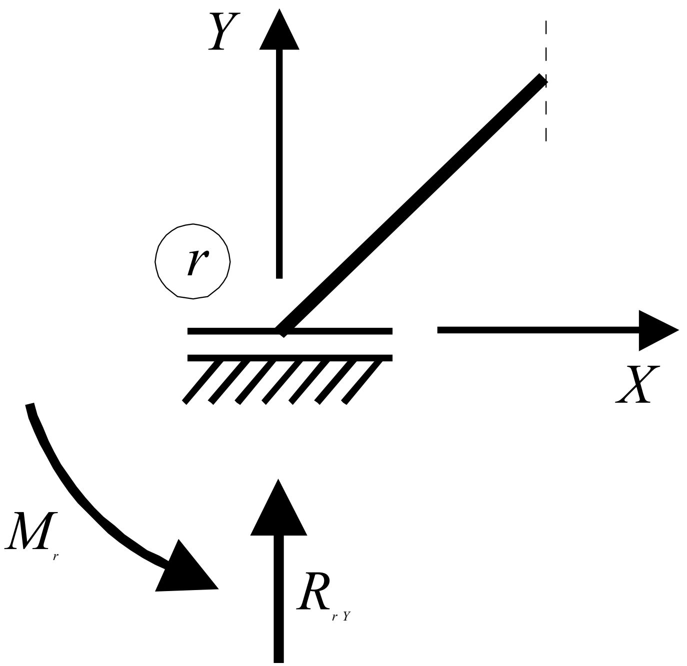
Boundary conditions of plane frame structures



$$u_{rX} = 0$$
$$u_{rY} = 0$$
$$\varphi_r = 0$$

Boundary conditions of plane frame structures

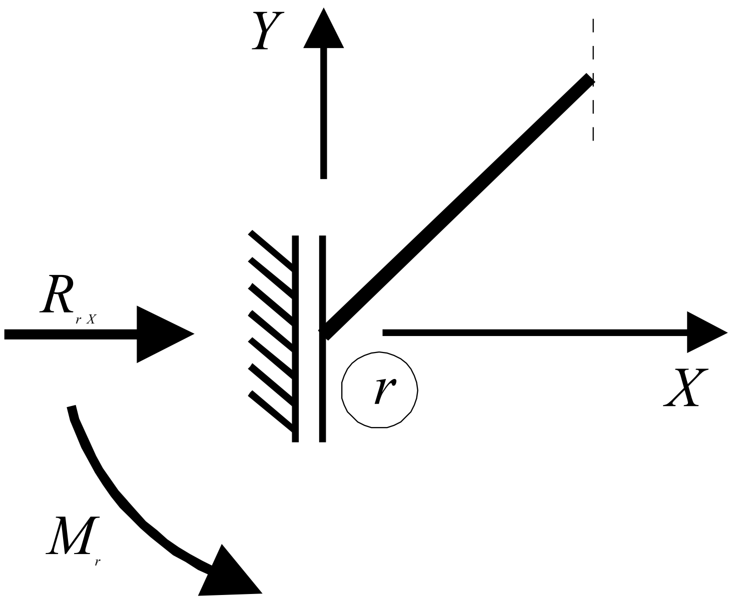
b) rigid-movable support (a displacement in the direction of the global X axis)



$$u_{rY} = 0$$
$$\varphi_r = 0$$

Boundary conditions of plane frame structures

c) rigid movable support (a displacement in the direction of the global *Y* axis)



$$u_{rX} = 0$$

$$\varphi_r = 0$$

Boundary conditions of plane frame structures

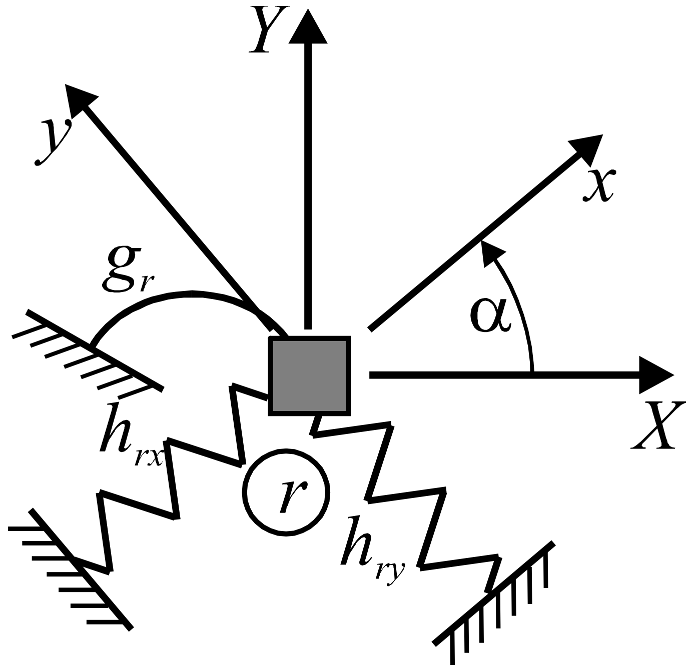
Considering boundary conditions requires the modification of a global stiffness matrix of a structure and it is done identically as for a plane truss, thus, we will not describe the way of modifying this matrix here. A whole range of other supports such as moveable skew supports and elastic supports considered analogously to supports of trusses described for plane truss is also possible.

Boundary elements of 2D frames

Introducing a boundary element is a convenient way to avoid problems connected with the consideration of different, non-typical boundary conditions. It allows, in fact, us to model fixed and fixed-movable supports with approximate exactness and to substitute elastic supports.

Boundary elements of 2D frames

Now we will present a single elastic support inclined at some angle. The scheme of this element and notations used are shown here:



Boundary elements of 2D frames

Stiffness of springs: h_{rx} and h_{ry} are forces which should be applied to their ends in order to induce unitary extensions. Rotation stiffness of a support g_r is a moment necessary to induce the rotation of the node r equal to one radian.

Boundary elements of 2D frames

The stiffness matrix of such an element in the local coordinate system has the form:

$$\mathbf{K}^b = \begin{bmatrix} h_{rx} & 0 & 0 \\ 0 & h_{ry} & 0 \\ 0 & 0 & g_r \end{bmatrix}$$

Boundary elements of 2D frames

Its transformation to the global system is done analogously to the case of normal frame or truss elements except that it concerns one node only $\mathbf{K}^e = \mathbf{R}^e \mathbf{K}'^e \left(\mathbf{R}^e\right)^{\mathrm{T}}$. The rotation matrix is given by $\frac{d^4 y}{dx^4} = \frac{q_y(x)}{EJ_z}$. Hence we can write the equation transforming the matrix to the global system:

$$\mathbf{K}^b = \mathbf{R}_r \mathbf{K}'^b \mathbf{R}_r^{\mathrm{T}}$$

Boundary elements of 2D frames

After taking into consideration $\dfrac{d^4 y}{dx^4} = \dfrac{q_y(x)}{EJ_z}$ and

$$\mathbf{K}^b = \begin{bmatrix} h_{rx} & 0 & 0 \\ 0 & h_{ry} & 0 \\ 0 & 0 & g_r \end{bmatrix}$$

we obtain:

$$\mathbf{K}^b = \begin{bmatrix} c^2 h_{rx} + s^2 h_{ry} & sc\left(h_{rx} - h_{ry}\right) & 0 \\ sc\left(h_{rx} - h_{ry}\right) & c^2 h_{rx} + s^2 h_{ry} & 0 \\ 0 & 0 & g_r \end{bmatrix}$$

$$s = \sin\alpha \qquad c = \cos\alpha$$

Boundary elements of 2D frames

If we model flexible supports we ought to assume high stiffness of a suitable spring. In most cases, stiffness of the order of $1 \cdot 10^{30}$ assures similarity between results obtained with this method and the results obtained with the exact method.

Internal forces due to a static load

The variety of loads which can act on a frame structure is greater than it was in the case of a truss. Frame elements can be affected by concentrated (forces, moments), distributed (pressure, moment loads) and temperature loads. The formulation of equilibrium equations requires substitution of internode loads for an equivalent set of concentrated forces and moments acting on nodes.

Internal forces due to a static load

Equation $y(x) = C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4$ define displacements of an element bending in the direction of the *y* axis of the global system. After adding the equations describing the displacements in an axial direction, we obtain relations defining the displacements vector for any point between nodes

$$\mathbf{u}(x) = \begin{bmatrix} u_x(x) \\ u_y(x) \\ \varphi(x) \end{bmatrix} = \mathbf{N}\,\mathbf{u}^e$$

Internal forces due to a static load

$$\mathbf{u}(x) = \begin{bmatrix} u_x(x) \\ u_y(x) \\ \varphi(x) \end{bmatrix} = \mathbf{N}\,\mathbf{u}^e$$

N - the rectangular matrix of shape functions. It contains two blocks:

N$_i$(x) - matrix of the shape functions for the first node,

N$_j$(x) - matrix of the shape functions for the last node.

$$\mathbf{N}(x) = \begin{bmatrix} \mathbf{N}_i(x) & \mathbf{N}_j(x) \end{bmatrix}$$

Internal forces due to a static load

We can obtain both matrices from equations

$y(x) = C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4$ and $\Delta L = u_{jx} - u_{ix}$:

$$\mathbf{N}_i(x) = \begin{bmatrix} \omega_1(\xi) & 0 & 0 \\ 0 & \omega_3(\xi) & L\omega_5(\xi) \\ 0 & \frac{1}{L}\omega_3{}'(\xi) & \omega_5{}'(\xi) \end{bmatrix} \qquad \mathbf{N}_j(x) = \begin{bmatrix} \omega_2(\xi) & 0 & 0 \\ 0 & \omega_4(\xi) & L\omega_6(\xi) \\ 0 & \frac{1}{L}\omega_4{}'(\xi) & \omega_6{}'(\xi) \end{bmatrix}$$

Internal forces due to a static load

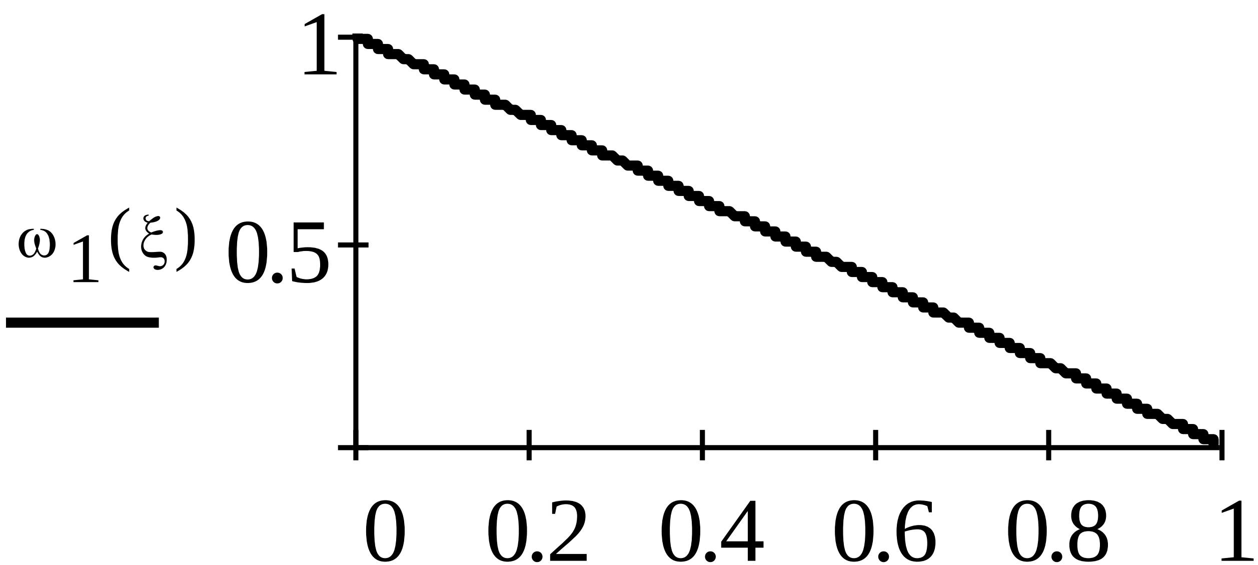
$$\mathbf{N}_i(x)=\begin{bmatrix}\omega_1(\xi) & 0 & 0\\ 0 & \omega_3(\xi) & L\omega_5(\xi)\\ 0 & \frac{1}{L}\omega_3{}'(\xi) & \omega_5{}'(\xi)\end{bmatrix}\qquad \mathbf{N}_j(x)=\begin{bmatrix}\omega_2(\xi) & 0 & 0\\ 0 & \omega_4(\xi) & L\omega_6(\xi)\\ 0 & \frac{1}{L}\omega_4{}'(\xi) & \omega_6{}'(\xi)\end{bmatrix}$$

The convenient non-dimensional coordinate

$\xi = x/L$ is introduced here. Non-dimensional displacement functions $\omega_i(\xi)$(i = 1,2 ... 6)) and their derivatives $\omega_i{}'(\xi)$, $\omega_i{}''(\xi)$ are surveyed on the following slides.

Internal forces due to a static load

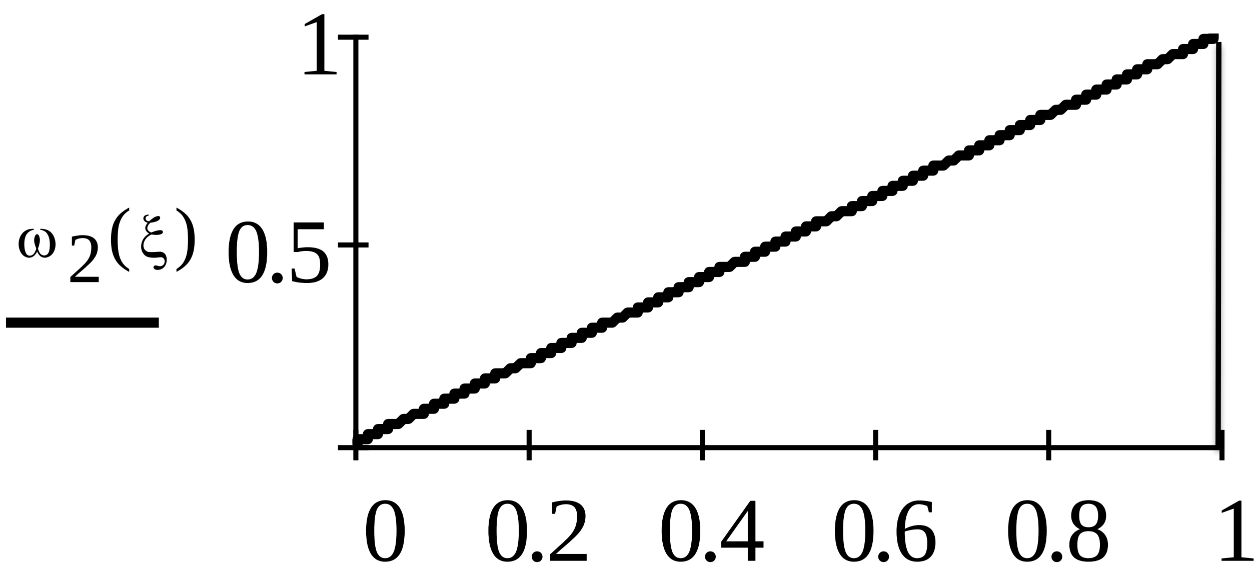
$\omega_1 = 1 - \xi$



Shape function of frame element with parallel displacement in node *i*.

Internal forces due to a static load

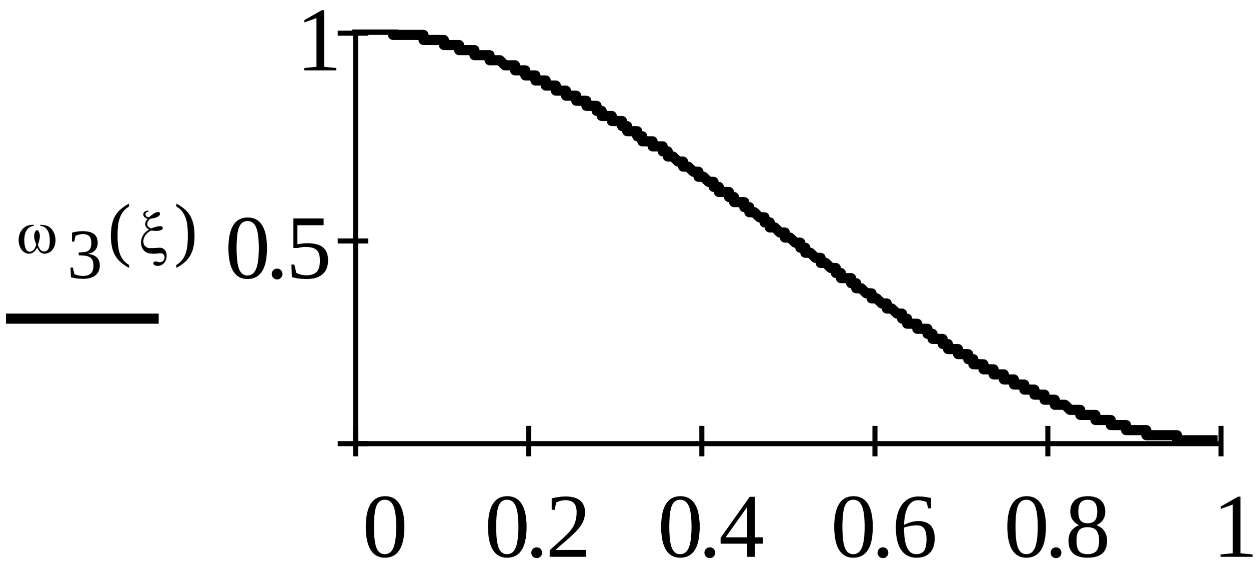
$\omega_2 = \xi$



Shape function of frame element with parallel displacement in node *j*.

Internal forces due to a static load

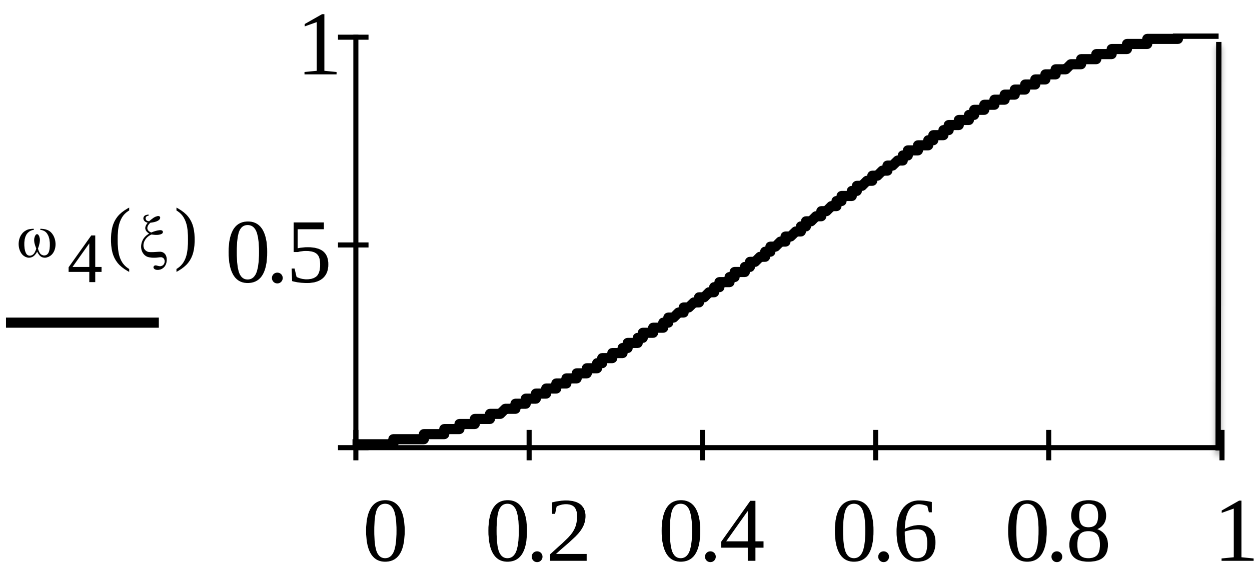
$\omega_3 = 1 - 3\xi^2 + 2\xi^3$



Shape function of frame element with perpendicular displacement in node *i*.

Internal forces due to a static load

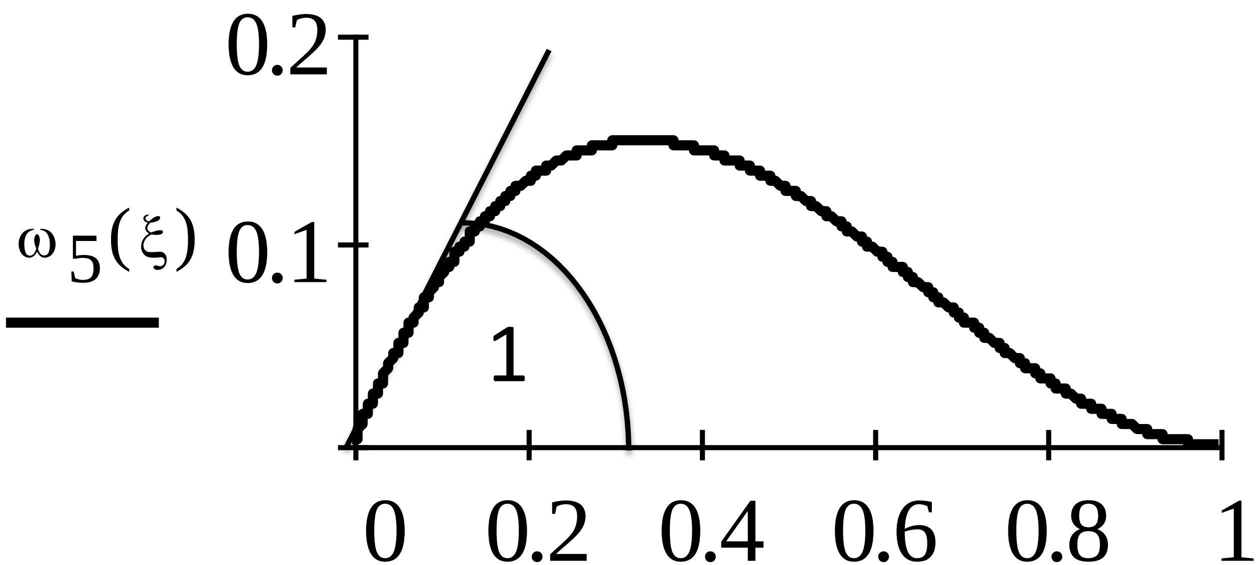
$\omega_4 = \xi^2(3 - 2\xi)$



Shape function of frame element with perpendicular displacement in node *j*.

Internal forces due to a static load

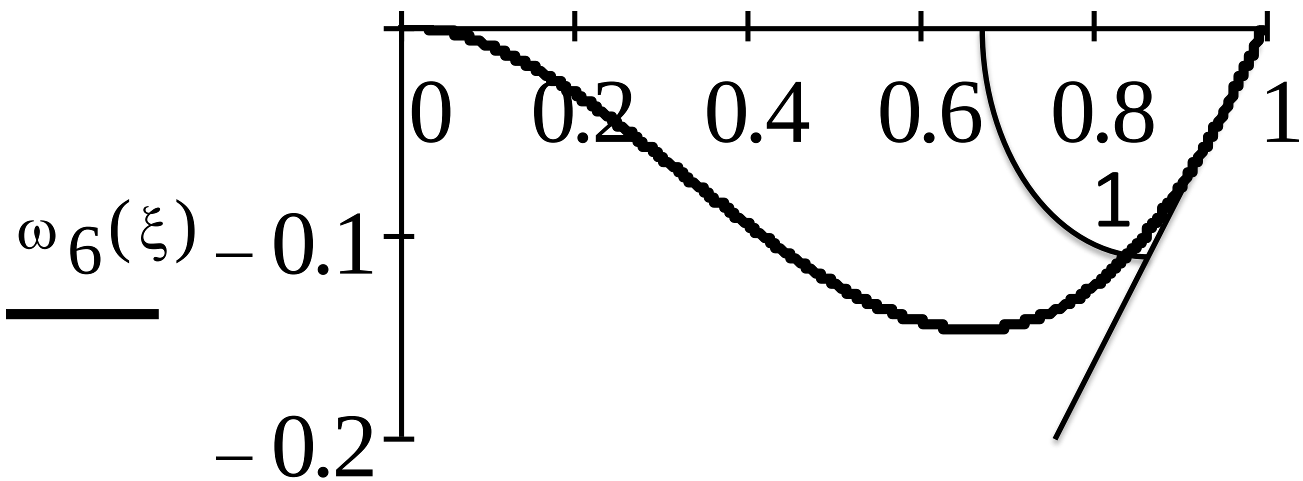
$\omega_5 = \xi(1 - 2\xi + \xi^3)$



Shape function of frame element with rotation in node *i*.

Internal forces due to a static load

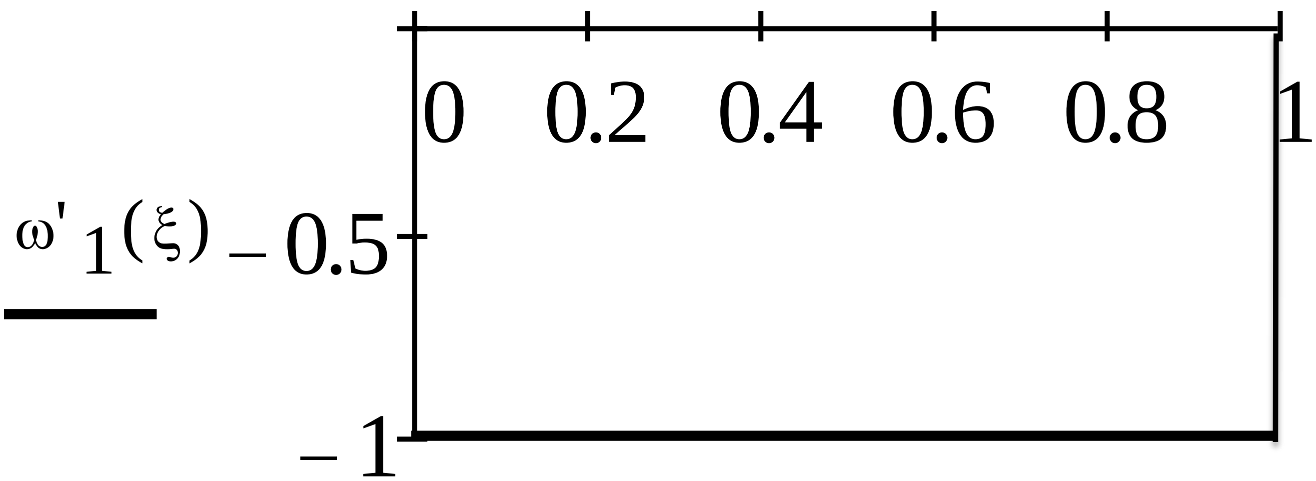
$\omega_6 = - \xi^2(1 - \xi)$



Shape function of frame element with rotation in node *j*.

Internal forces due to a static load

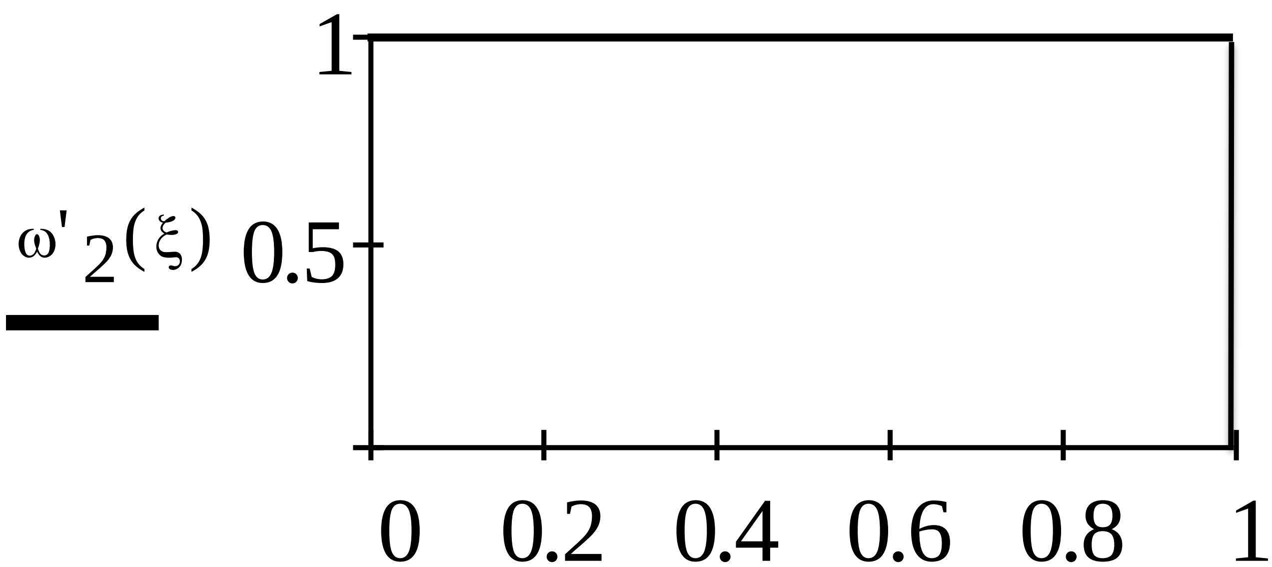
$\omega'_1 = -1$



The rotation function of frame element with parallel displacement in node *i.*

Internal forces due to a static load

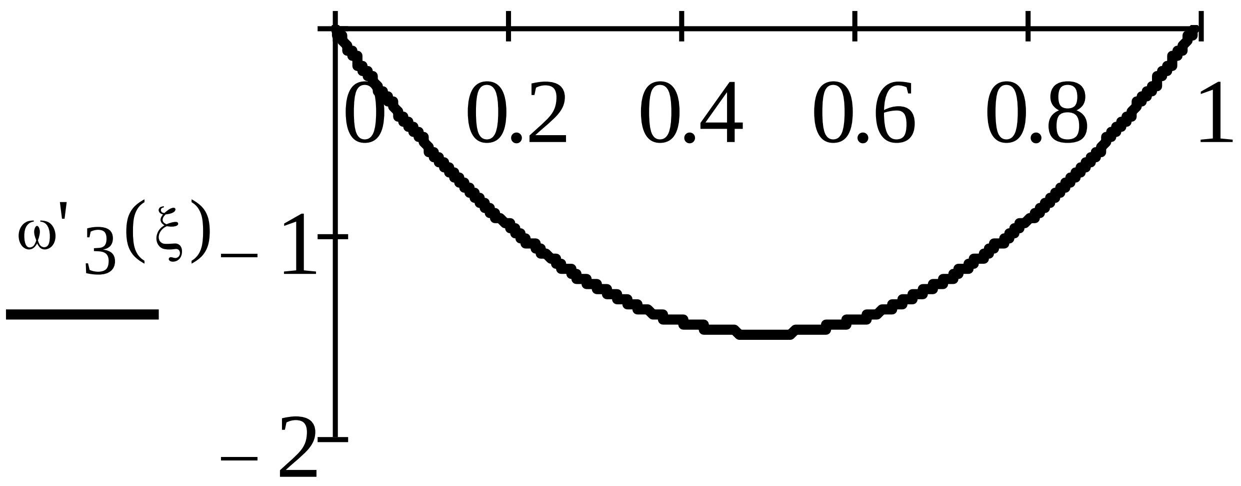
$\omega'_2 = 1$



The rotation function of frame element with parallel displacement in node *j*.

Internal forces due to a static load

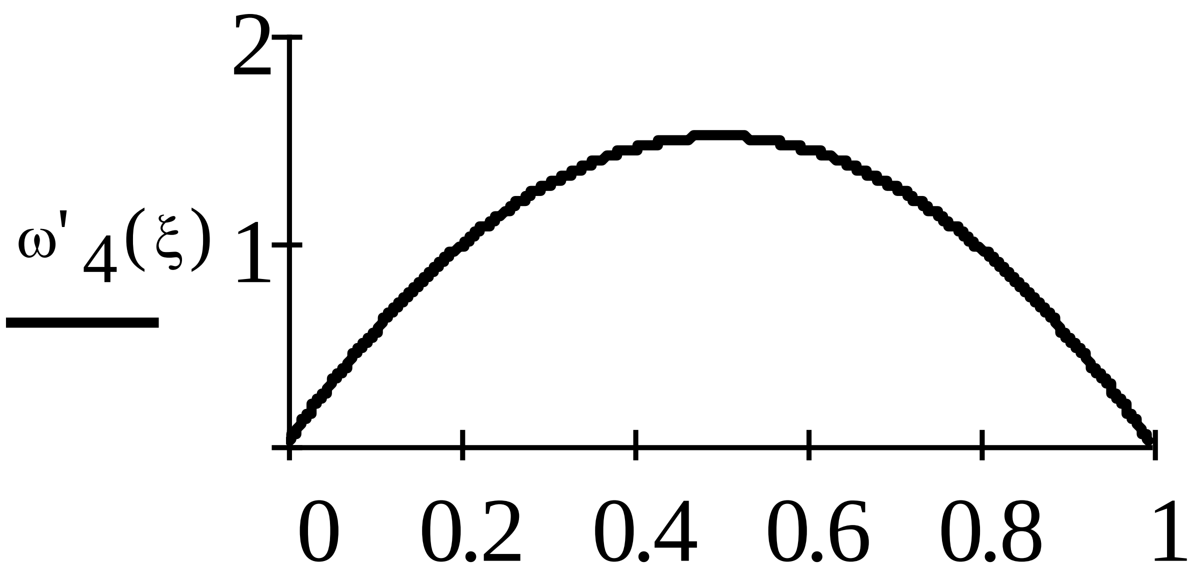
$\omega'_3 = -6\xi(1-\xi)$



The rotation function of frame element with perpendicular displacement in node *i.*

Internal forces due to a static load

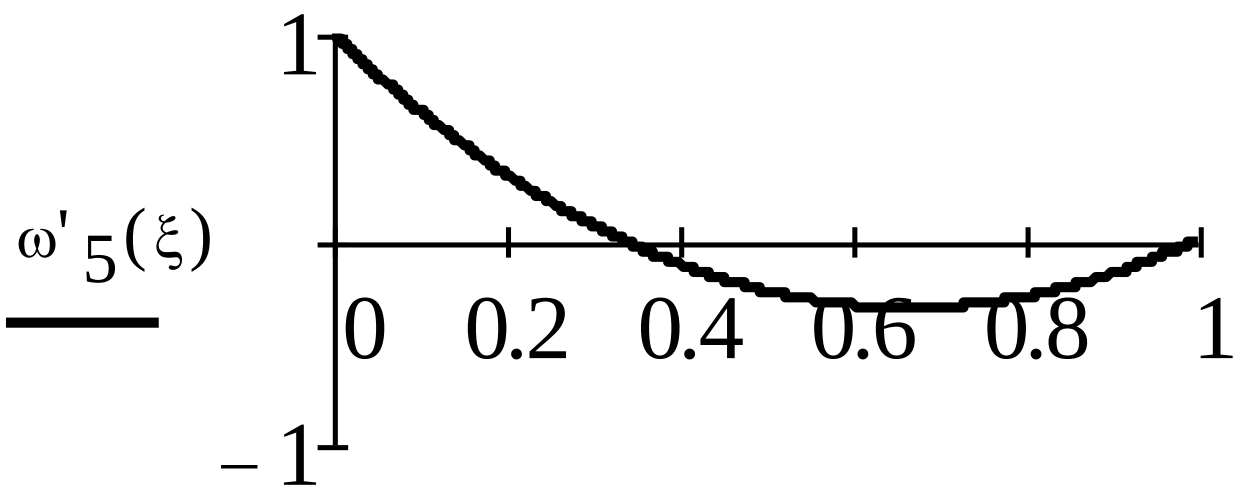
$\omega'_4 = 6\xi(1 - \xi)$



The rotation function of frame element with perpendicular displacement in node *j*.

Internal forces due to a static load

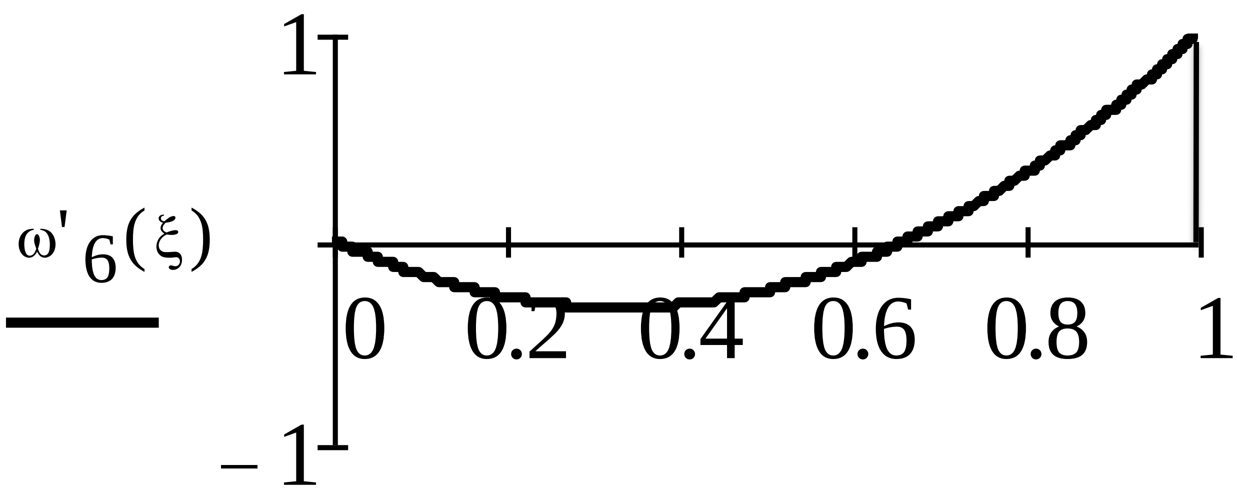
$\omega'_5 = 1 - 4\xi + 3\xi^2$



The rotation function of frame element with rotation in node *i*.

Internal forces due to a static load

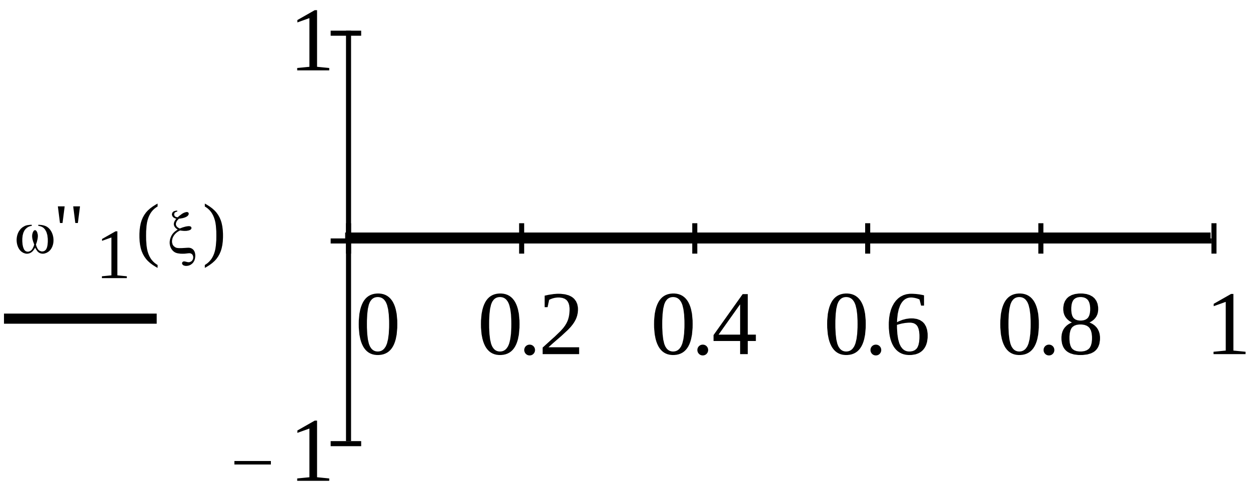
$\omega'_6 = -\xi(2 - 3\xi)$



The rotation function of frame element with rotation in node *j*.

Internal forces due to a static load

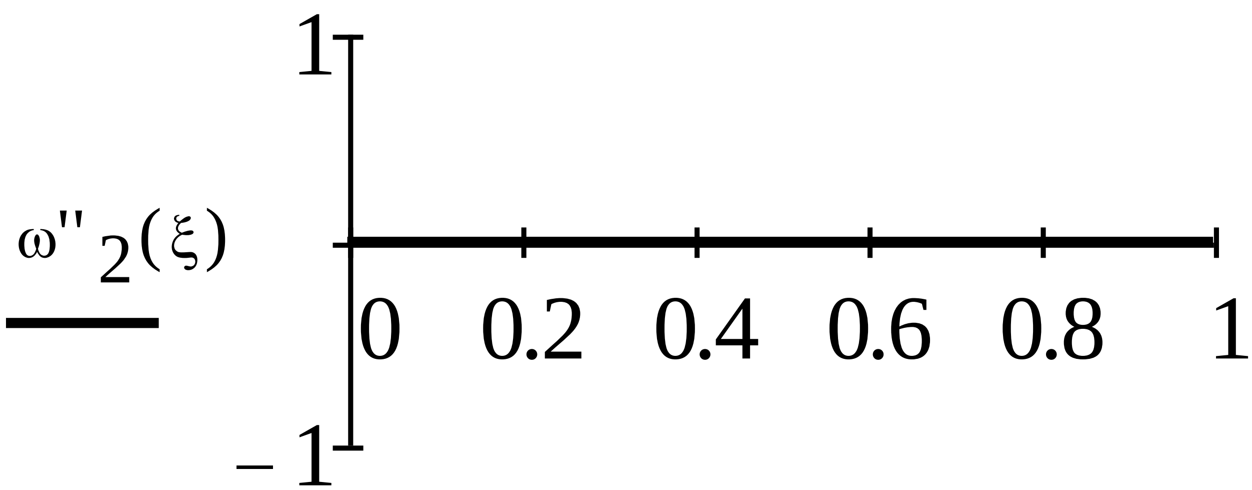
$\omega''_1 = 0$



The bending moment function of frame element with parallel displacement in node *i*.

Internal forces due to a static load

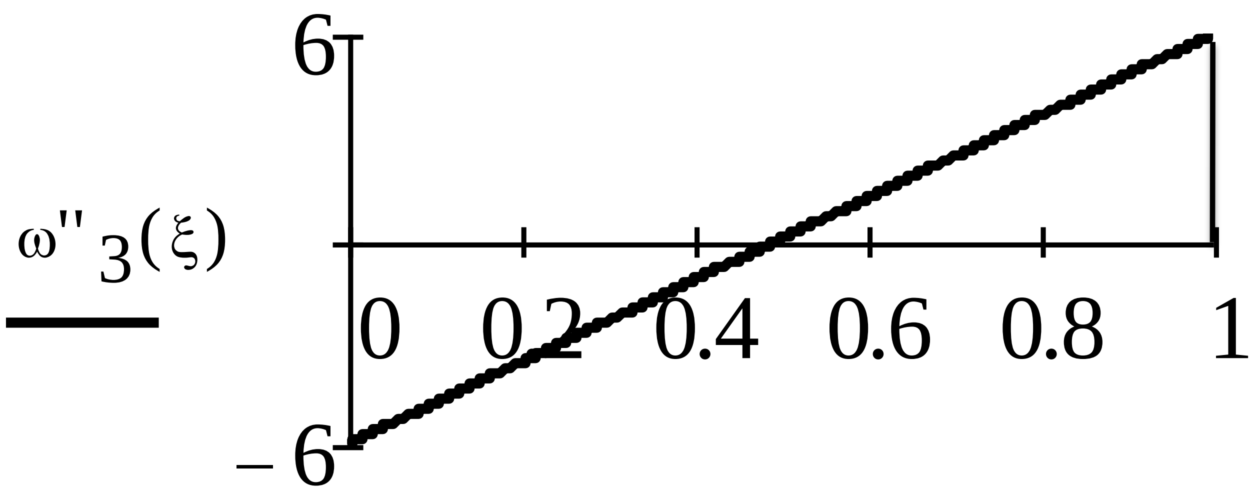
$\omega''_2 = 0$



The bending moment function of frame element with parallel displacement in node *j*.

Internal forces due to a static load

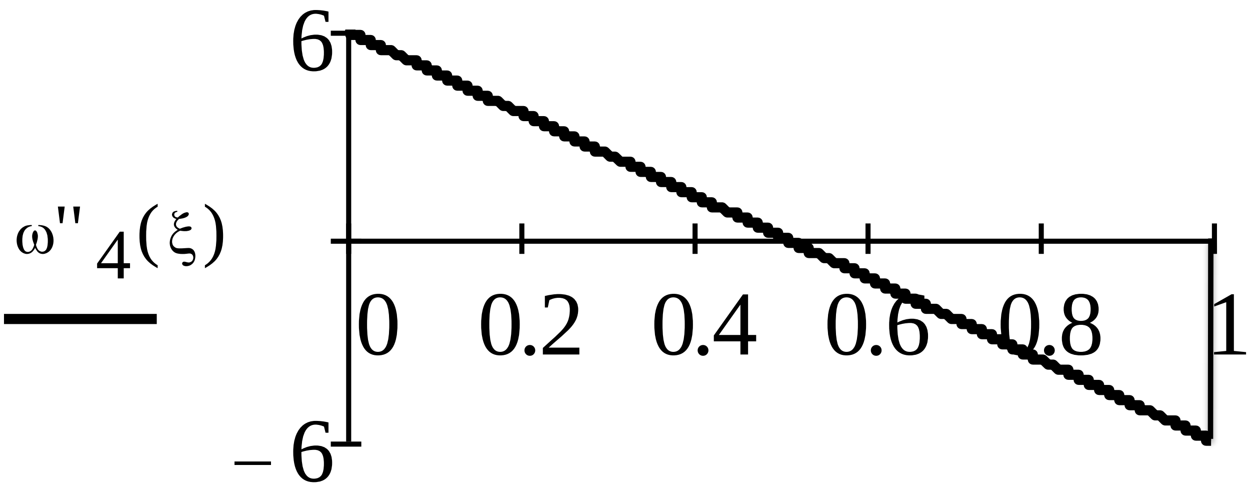
$\omega''_3 = -6 + 12\xi$



The bending moment function of frame element with perpendicular displacement in node *i.*

Internal forces due to a static load

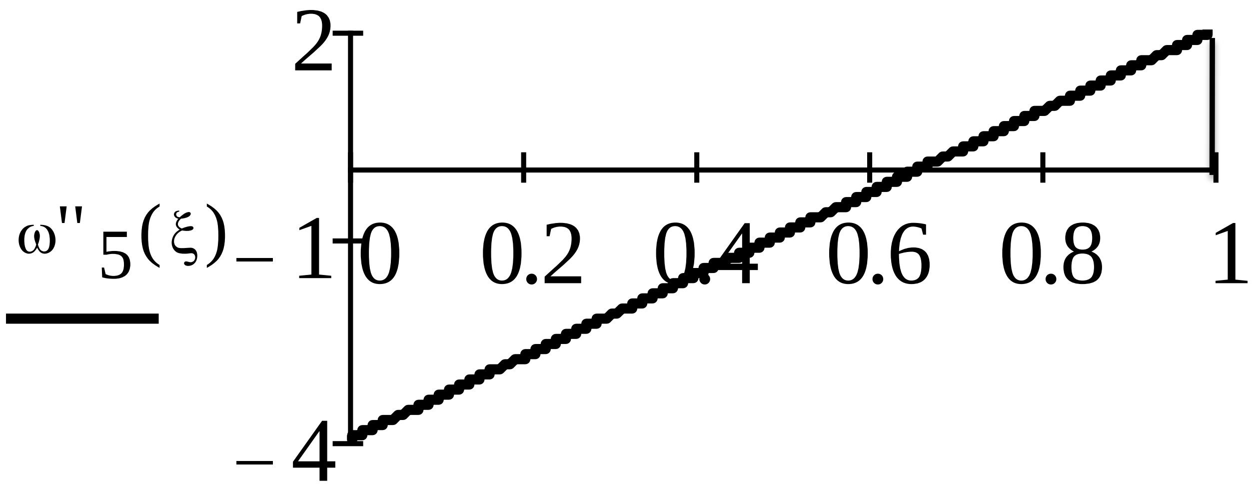
$\omega''_4 = 6 - 12\xi$



The bending moment function of frame element with perpendicular displacement in node *j*.

Internal forces due to a static load

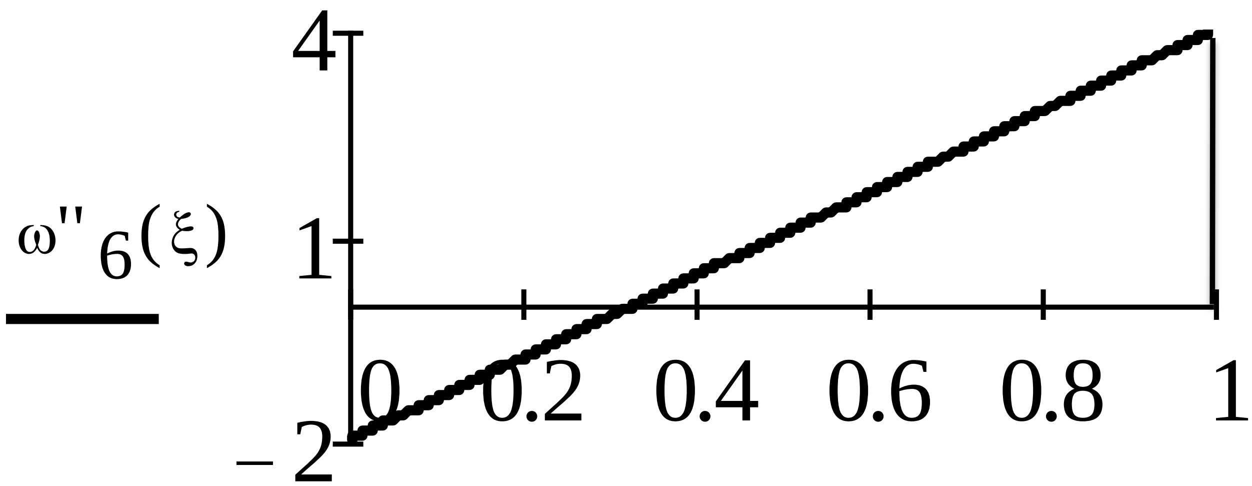
$\omega''_5 = - 4 + 6\xi$



The bending moment function of frame element with rotation in node *i.*

Internal forces due to a static load

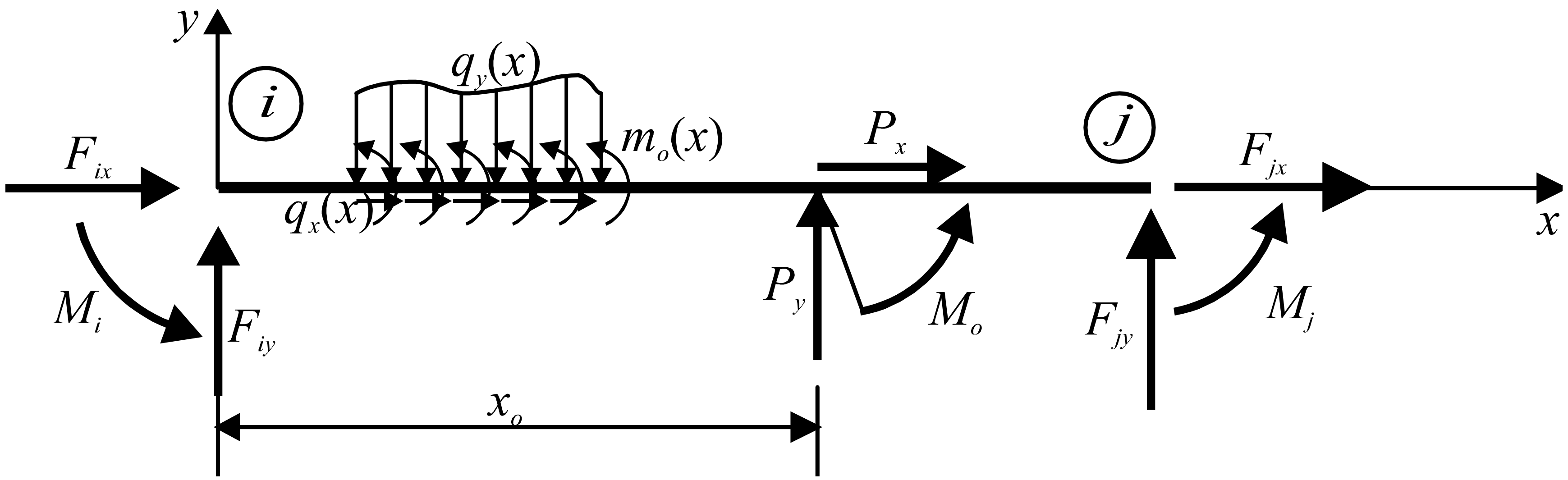
$\omega''_6 = -2 + 6\xi$



The bending moment function of frame element with rotation in node *j*.

Internal forces due to a static load

Let us consider now the bar (an element) of a plane frame loaded with static loads



Internal forces due to a static load

We will find nodal forces $\mathbf{f}^{e}$ by making use of conditions of element equilibrium. We will use the principle of virtual work here:

$$L_n = \left(\mathbf{f}'^{e}\right)^{\mathrm{T}} \mathbf{u}^{e}$$

- the work of nodal forces,

$$L_z = \int_0^L \left[q_y(x)\, u_y(x) + q_x(x)\, u_x(x) + m_o(x)\, \varphi(x) \right] dx$$

- the work of external forces (static loads).

Internal forces due to a static load

Concentrated forces and moments can also be analysed by describing them in the following way:

$$q(x) = \delta(x - x_o)P$$

$$M_o = \delta(x - x_o)M_o$$

where $\delta(x_o)$ is Dirac's delta.

Internal forces due to a static load

The Dirac's delta $\delta(x_o)$ is defined as:

$$\delta(x - x_o) = 0, \quad \text{while } x<x_0;$$

$$\delta(x - x_o) \to \infty, \quad \text{while } x=x_0;$$

$$\delta(x - x_o) = 0, \quad \text{while } x>x_0;$$

Internal forces due to a static load

The element equilibrium is maintain when

$L_n + L_z = 0$, which means:

$$\left(\mathbf{f}'^{e}\right)^{\mathrm{T}} \mathbf{u}^{e} = -\int_{0}^{L} \left[\mathbf{q}(x)\right]^{\mathrm{T}} \mathbf{u}(x) dx$$

where **q**(*x*) is the vector of external loads:

$$\mathbf{q}(x) = \begin{bmatrix} q_x(x) \\ q_y(x) \\ m_o(x) \end{bmatrix}$$

Internal forces due to a static load

Putting the expression describing the element displacements vector

$$\mathbf{u}(x) = \begin{bmatrix} u_x(x) \\ u_y(x) \\ \varphi(x) \end{bmatrix} = \mathbf{N}\,\mathbf{u}^e$$ into $$\left(\mathbf{f}'^e\right)^{\mathrm{T}}\mathbf{u}^e = -\int_0^L [\mathbf{q}(x)]^{\mathrm{T}}\mathbf{u}(x)dx$$

we obtain relations:

$$\left(\mathbf{f}'^e\right)^{\mathrm{T}}\mathbf{u}^e = -\int_0^L \mathbf{q}^{\mathrm{T}}\,\mathbf{N}\,\mathbf{u}^e\,dx \qquad \left(\mathbf{f}'^e\right)^{\mathrm{T}} = -\int_0^L \mathbf{N}\,\mathbf{q}\,dx$$

Internal forces due to a static load

These relations enables us to replace loads acting on elements by loads acting on nodes. It should be noted here that there are forces acting on the nodes in the equilibrium equations and that these forces act against those acting on the element thus, they should be subtracted from the nodal forces vector of the structure.

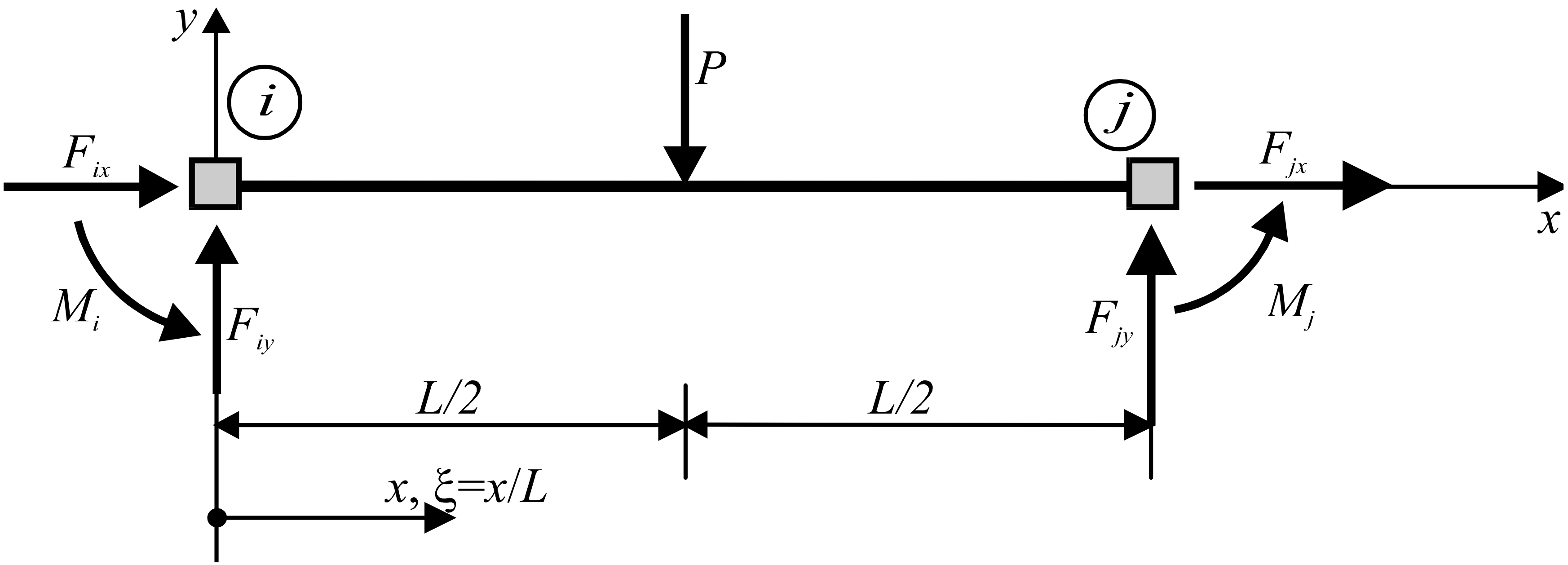
Internal forces due to a static load

We check the effectiveness of equation $\left(\mathbf{f}'^{e}\right)^{\mathrm{T}} = -\int_{0}^{L} \mathbf{N}\,\mathbf{q}\,dx$ for three simple examples when:

- the load with a concentrated force is applied to the centre of an element,
- the load with a concentrated moment,
- the distributed load which is constant for the whole element.

Internal forces due to a static load

Example No 1.



The frame element loaded with a concentrated force.

Internal forces due to a static load

We introduce a non-dimensional coordinate to make the calculations more convenient and write the concentrated force as follows:

$$\mathbf{q}(\xi) = \begin{bmatrix} 0 \\ P\delta(\xi - 0.5) \\ 0 \end{bmatrix}$$

and after putting it into Eqn. $\left(\mathbf{f}'^{e}\right)^{\mathrm{T}} = -\int_0^L \mathbf{N}\,\mathbf{q}\,dx$ we obtain:

Internal forces due to a static load

$$\mathbf{f}'^{e} = -L^2 \int_0^1 \begin{bmatrix} (\mathbf{N}_i)^{\mathrm{T}} \\ (\mathbf{N}_j)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} 0 \\ -P\delta(\xi - 0.5) \\ 0 \end{bmatrix} d\xi = P\int_0^1 \begin{bmatrix} 0 \\ \delta(\xi-0.5)\omega_3(\xi) \\ L\delta(\xi-0.5)\omega_5(\xi) \\ 0 \\ \delta(\xi-0.5)\omega_4(\xi) \\ L\delta(\xi-0.5)\omega_6(\xi) \end{bmatrix} d\xi =$$

$$= P\begin{bmatrix} 0 \\ \omega_3(0.5) \\ L\omega_5(0.5) \\ 0 \\ \omega_4(0.5) \\ L\omega_6(0.5) \end{bmatrix} = P\begin{bmatrix} 0 \\ 0.5 \\ L/8 \\ 0 \\ 0.5 \\ -L/8 \end{bmatrix}$$

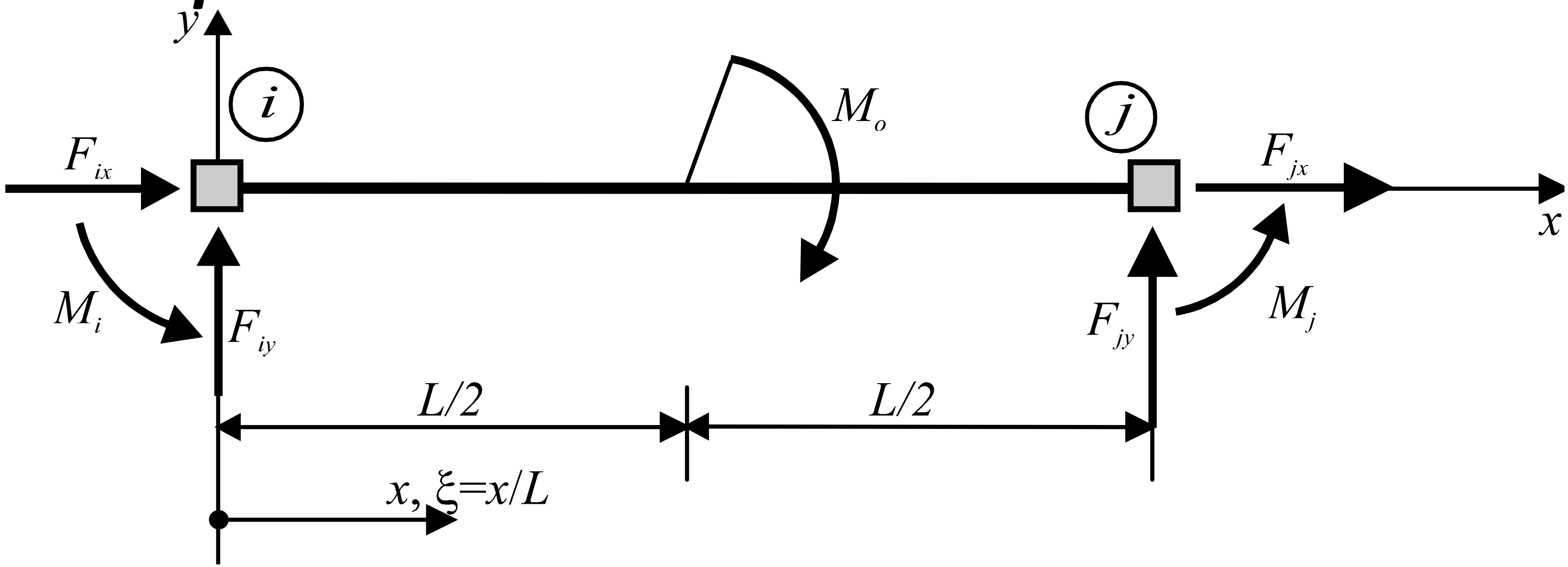
which means that

$$F_{ix} = 0 \qquad F_{iy} = \frac{1}{2}P \qquad M_i = \frac{1}{8}PL$$

$$F_{jx} = 0 \qquad F_{jy} = \frac{1}{2}P \qquad M_j = -\frac{1}{8}PL$$

Internal forces due to a static load

Example No 2.



The frame element loaded with a concentrated moment.

Internal forces due to a static load

We write the concentrated moment applied to the centre of an element by using Dirac's delta:

$$\mathbf{q}(\xi) = \begin{bmatrix} 0 \\ 0 \\ -\dfrac{M_o}{L}\delta(\xi - 0.5) \end{bmatrix}$$

and after putting it into Eqn. $\left(\mathbf{f}'^{e}\right)^{\mathrm{T}} = -\int_0^L \mathbf{N}\,\mathbf{q}\,dx$ we obtain:

Internal forces due to a static load

$$\mathbf{f}'^{e} = \frac{M_o}{L}\int_0^1 \begin{bmatrix} (\mathbf{N}_i)^{\mathrm{T}} \\ (\mathbf{N}_j)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \delta(\xi - 0.5) \end{bmatrix} d\xi = \frac{M_o}{L}\int_0^1 \begin{bmatrix} 0 \\ \omega_3'(\xi)\delta(\xi-0.5)/L \\ \omega_5'(\xi)\delta(\xi-0.5) \\ 0 \\ \omega_4'(\xi)\delta(\xi-0.5)/L \\ \omega_6'(\xi)\delta(\xi-0.5) \end{bmatrix} d\xi = M_o \begin{bmatrix} 0 \\ -\dfrac{3}{2L} \\ -\dfrac{1}{4} \\ 0 \\ \dfrac{3}{2L} \\ -\dfrac{1}{4} \end{bmatrix}$$

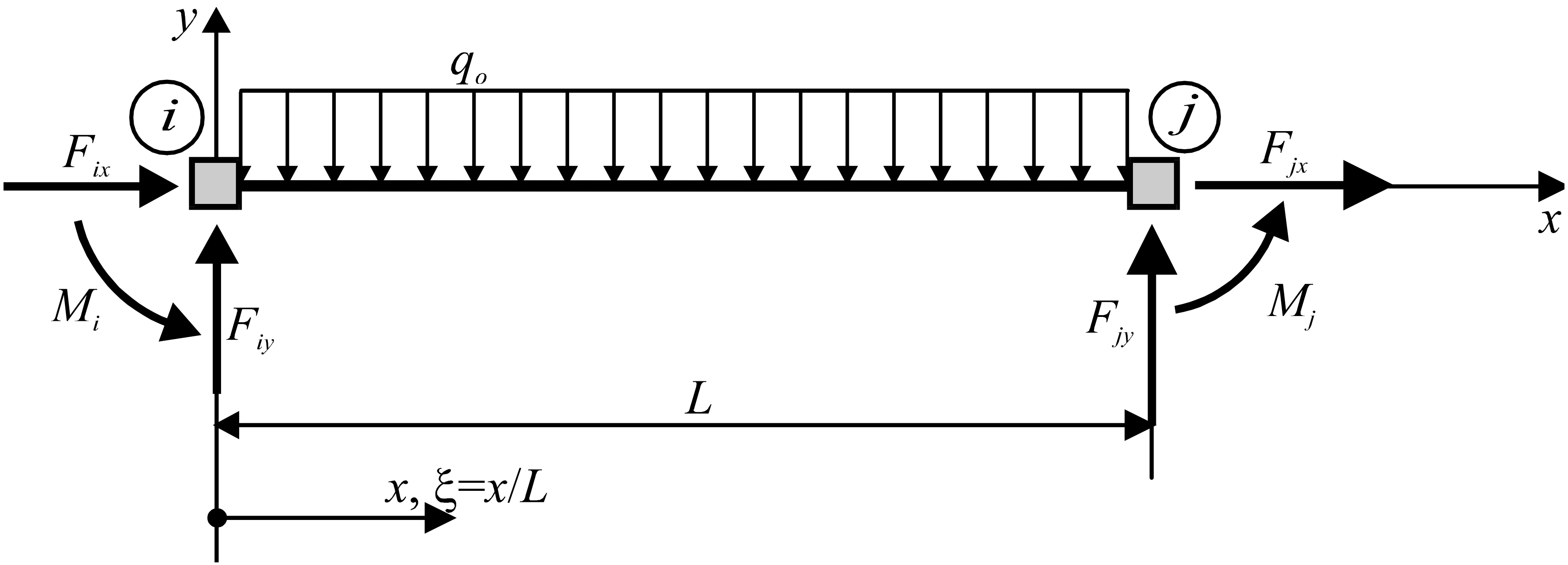
which means that

$$F_{ix} = 0 \quad F_{iy} = -\frac{3}{2L}M_o \qquad M_i = -\frac{1}{4}M_o$$

$$F_{jx} = 0 \quad F_{jy} = \frac{3}{2L}M_o \qquad M_j = -\frac{1}{4}M_o$$

Internal forces due to a static load

Example No 3.



The frame element loaded with a uniformly distributed load.

Internal forces due to a static load

The continuous load uniformly distributed on the whole length of an element gives a load vector:

$$\mathbf{q}(\xi) = q_o \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

and after putting it into Eqn. $\left(\mathbf{f}'^{e}\right)^{\mathrm{T}} = -\int_0^L \mathbf{N}\,\mathbf{q}\,dx$ we obtain:

Internal forces due to a static load

$$\mathbf{f}'^{e} = q_o L \int_0^1 \begin{bmatrix} 0 \\ \omega_3(\xi) \\ L\omega_5(\xi) \\ 0 \\ \omega_4(\xi) \\ L\omega_6(\xi) \end{bmatrix} d\xi = q_o L \begin{bmatrix} 0 \\ 1/2 \\ L/12 \\ 0 \\ 1/2 \\ -L/12 \end{bmatrix}$$

which means that

$$F_{ix} = 0 \quad F_{iy} = \frac{1}{2} q_o L \quad M_i = \frac{1}{12} q_o L$$

$$F_{jx} = 0 \quad F_{jy} = \frac{1}{2} q_o L \quad M_j = -\frac{1}{12} q_o L$$

Forces caused by a temperature load

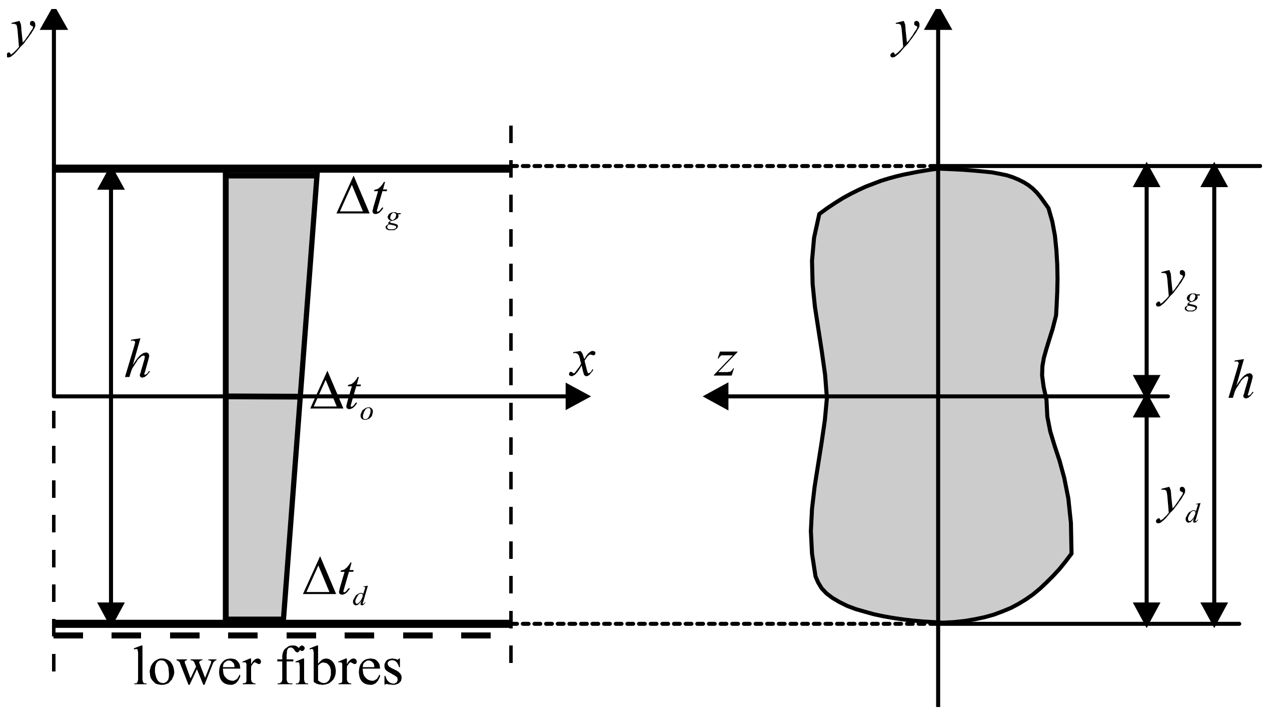
The action of a temperature on frame elements can cause flexion. This happens when the temperature field is not homogeneous in the cross section. In the case of a truss, the flexion of bars did not cause increasing nodal forces because truss elements are connected by means of jointed nodes.

Forces caused by a temperature load

Bars of frame structures can make a node rotate, hence we have to determine forces at the node in the element undergoing the action of the non-uniform temperature field.

Let us consider an element of which the upper fibres are affected by an increase in a temperature Δt_g, and the lower fibres are affected by an increase in a temperature Δt_d.

Forces caused by a temperature load



The temperature distribution in the element cross section.

Forces caused by a temperature load

The temperature field can be written as follows:

$$\Delta t(x,y) = \Delta t_o(x) + \frac{y}{h}\Delta t_h(x)$$

$\Delta t_o = \frac{1}{h}\left[\Delta t_d y_g + \Delta t_g y_d\right]$ - the increase in the temperature of the middle fibres,

$\Delta t_h = \Delta t_g - \Delta t_d$ - the difference of temperatures between extreme fibres,

h - is the height of the cross section,

Forces caused by a temperature load

$$\Delta t(x,y) = \Delta t_o(x) + \frac{y}{h}\Delta t_h(x)$$

y_d - the distance between the centre of gravity and the lower fibres,

y_g - is the distance between the centre of gravity and the upper fibres.

Forces caused by a temperature load

Strains of the element fibres induced by the temperature field are equal to

$$\varepsilon_t(y) = \alpha_t \Delta t(y) = \alpha_t \left(\Delta t_o + \Delta t_h \frac{y}{h} \right)$$

where α_t is the expansion coefficient of the material.

Forces caused by a temperature load

If bars cannot deform freely, then stresses rise inside them:

$$\sigma_x = -E\varepsilon_t = -\alpha_t E\left(\Delta t_o + \Delta t_h \frac{y}{h}\right)$$

which the internal forces result from:

$$N = \int_A \sigma_x dA = -\alpha_t E\left(\Delta t_o \int_A dA + \frac{\Delta t_h}{h}\int_A y dA\right)$$

Forces caused by a temperature load

Since the second integral occurring in previous equation is the static moment with regard to the *z* axis which crosses the centre of gravity, this moment has to be equal to zero. Thus, we obtain

$$N_t(x) = -\alpha_t \Delta t_o(x) EA$$

like in the case of a truss element.

Forces caused by a temperature load

The second internal force caused by temperature stresses is the bending moment:

$$M_t(x) = \int_A -\sigma_x(x) y dA = \alpha_t E \left[\Delta t_o(x) \int_A y dA + \frac{\Delta t_h}{h} \int_A y^2 dA \right]$$

The first integral has to be equal to zero similarly to $N = \int_A \sigma_x dA = -\alpha_t E \left(\Delta t_o \int_A dA + \frac{\Delta t_h}{h} \int_A y dA \right)$ and the second one is the moment of inertia of the cross section calculated with regard to the middle axis.

Forces caused by a temperature load

Thus, we can write an equation describing the bending moment due to temperature stresses as

$$M_t(x) = \frac{\alpha_t \Delta t_h(x)}{h} EJ_z$$

where $J_z = \int_A y^2 dA$ is the moment of inertia of the element section with regard to the *z* axis crossing the centre of gravity of the section.

Forces caused by a temperature load

We calculate forces at nodes making use of the principle of virtual work just as we did before in this presentation:

$$\left(\mathbf{u}^e\right)^{\mathrm{T}}\mathbf{f}'^{et} = \int_0^L \left[\boldsymbol{\varepsilon}(x)\right]^{\mathrm{T}}\mathbf{t}_t\,dx$$

where

$$\mathbf{t}_t = \begin{bmatrix} N_t(x) \\ 0 \\ M_t(x) \end{bmatrix}$$

Forces caused by a temperature load

$\mathbf{t}_t$ is the vector of the internal forces induced by a temperature. The zero value of the expression in the second row of the vector comes from the fact that the temperature does not cause shearing forces in the elements, $\boldsymbol{\varepsilon}(x)$ is the vector of displacements gradients:

$$\boldsymbol{\varepsilon}(x) = \begin{bmatrix} \frac{du_x}{dx} \\ \frac{du_y}{dy} \\ \frac{d\varphi}{dx} \end{bmatrix} = \mathbf{B}\mathbf{u}^e$$

Forces caused by a temperature load

$$\boldsymbol{\varepsilon}(x) = \mathbf{B}\mathbf{u}^e$$

B is the matrix of derivatives of shape functions:

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_i & \mathbf{B}_j \end{bmatrix}.$$

On the basis of

$$\mathbf{N}_i(x) = \begin{bmatrix} \omega_1(\xi) & 0 & 0 \\ 0 & \omega_3(\xi) & L\omega_5(\xi) \\ 0 & \frac{1}{L}\omega_3{}'(\xi) & \omega_5{}'(\xi) \end{bmatrix} \quad \mathbf{N}_j(x) = \begin{bmatrix} \omega_2(\xi) & 0 & 0 \\ 0 & \omega_4(\xi) & L\omega_6(\xi) \\ 0 & \frac{1}{L}\omega_4{}'(\xi) & \omega_6{}'(\xi) \end{bmatrix}$$

we calculate:

Forces caused by a temperature load

$$\mathbf{B}_i(x)=\begin{bmatrix} \frac{1}{L}\omega_1'(\xi) & 0 & 0 \\ 0 & \frac{1}{L}\omega_3'(\xi) & \omega_5'(\xi) \\ 0 & \frac{1}{L^2}\omega_3''(\xi) & \frac{1}{L}\omega_5''(\xi) \end{bmatrix}$$

$$\mathbf{B}_j(x)=\begin{bmatrix} \frac{1}{L}\omega_2'(\xi) & 0 & 0 \\ 0 & \frac{1}{L}\omega_4'(\xi) & \omega_6'(\xi) \\ 0 & \frac{1}{L^2}\omega_4''(\xi) & \frac{1}{L}\omega_6''(\xi) \end{bmatrix}$$

$\omega_i(\xi)$, $\omega_i'(\xi)$, $\omega_i''(\xi)$ (i = 1,2 ... 6)) are nondimensional functions given before in this presentation.

Forces caused by a temperature load

On the basis of $\left(\mathbf{u}^{e}\right)^{\mathrm{T}}\mathbf{f}'^{et} = \int_{0}^{L}\left[\boldsymbol{\varepsilon}(x)\right]^{\mathrm{T}}\mathbf{t}_{t}\,dx$ we calculate components of the nodal forces vector:

$$\mathbf{f}'^{et} = \int_{0}^{L}\mathbf{B}^{\mathrm{T}}\,\mathbf{t}_{t}\,dx$$

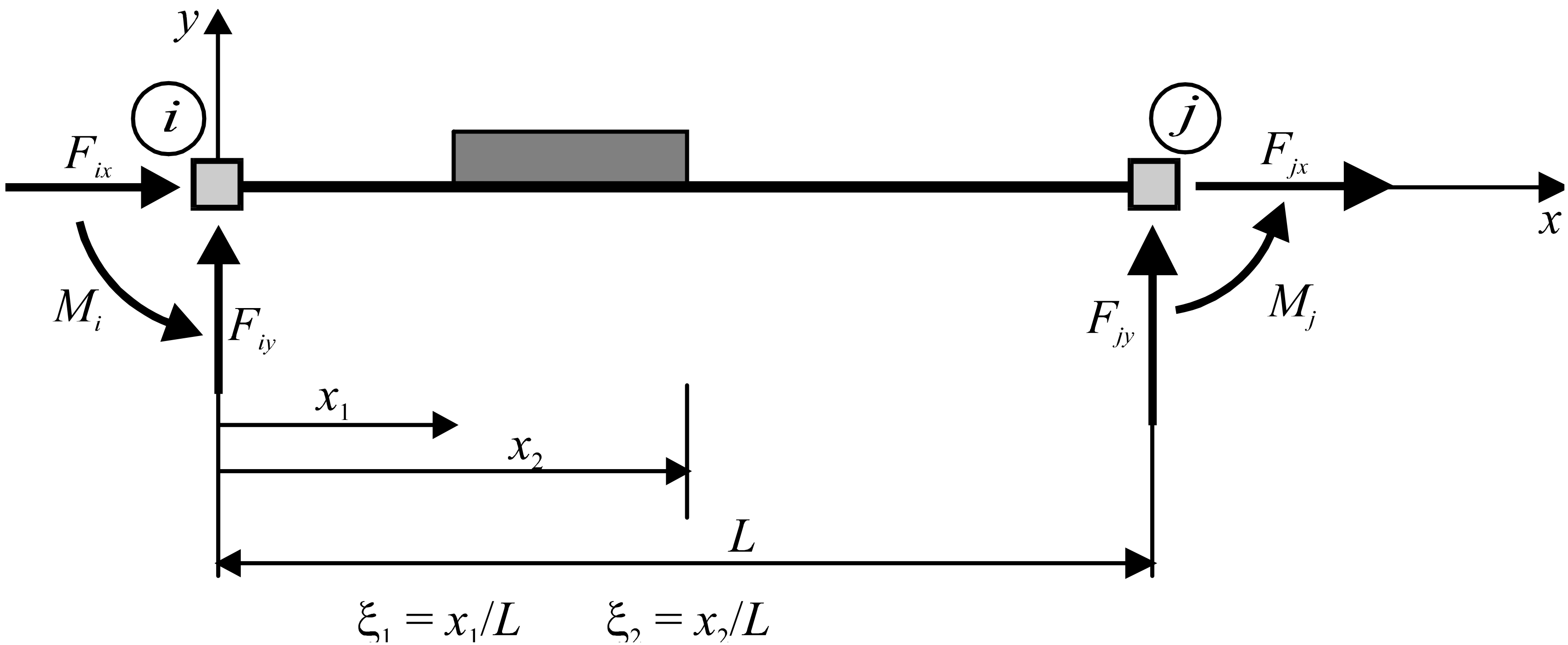
After inserting matrices $\mathbf{B}_i(x)$ and $\mathbf{B}_j(x)$ into equation $\mathbf{f}'^{et} = \int_{0}^{L}\mathbf{B}^{\mathrm{T}}\,\mathbf{t}_{t}\,dx$, we obtain

Forces caused by a temperature load

$$\mathbf{f}'^{et} = \alpha_t E \begin{bmatrix} -A\int_{\xi_1}^{\xi_2} \omega_1'(\xi)\Delta t_o(\xi)d\xi \\ \frac{J_z}{hL}\int_{\xi_1}^{\xi_2} \omega_3''(\xi)\Delta t_h(\xi)d\xi \\ \frac{J_z}{h}\int_{\xi_1}^{\xi_2} \omega_5''(\xi)\Delta t_h(\xi)d\xi \\ -A\int_{\xi_1}^{\xi_2} \omega_2'(\xi)\Delta t_o(\xi)d\xi \\ \frac{J_z}{hL}\int_{\xi_1}^{\xi_2} \omega_4''(\xi)\Delta t_h(\xi)d\xi \\ \frac{J_z}{h}\int_{\xi_1}^{\xi_2} \omega_6''(\xi)\Delta t_h(\xi)d\xi \end{bmatrix}$$

where ξ_1 and ξ_2 are non-dimensional coordinates at both the beginning and end of the action interval of the temperature load (following figure).

Forces caused by a temperature load



The temperature loaded frame element.

Forces caused by a temperature load

In the case when the temperature load is constant and occurs along the whole length of the element, we obtain the following equation from equations of $\mathbf{N}_i(x)$ and $\mathbf{N}_j(x)$:

$$\mathbf{f}'^{et} = \alpha_t E \begin{bmatrix} A\Delta t_o \\ 0 \\ -\dfrac{J_z \Delta t_h}{h} \\ -A\Delta t_o \\ 0 \\ \dfrac{J_z \Delta t_h}{h} \end{bmatrix}$$

Forces caused by a temperature load

$$L_n = \left(\mathbf{f}'^{e}\right)^{\mathrm{T}} \mathbf{u}^e \;, \qquad L_z = \int_0^L \left[q_y(x)\, u_y(x) + q_x(x)\, u_x(x) + m_o(x)\, \varphi(x) \right] dx$$

$$\mathbf{N}_i(x) = \begin{bmatrix} \omega_1(\xi) & 0 & 0 \\ 0 & \omega_3(\xi) & L\omega_5(\xi) \\ 0 & \frac{1}{L}\omega_3{}'(\xi) & \omega_5{}'(\xi) \end{bmatrix} \text{ and } \mathbf{N}_j(x) = \begin{bmatrix} \omega_2(\xi) & 0 & 0 \\ 0 & \omega_4(\xi) & L\omega_6(\xi) \\ 0 & \frac{1}{L}\omega_4{}'(\xi) & \omega_6{}'(\xi) \end{bmatrix}$$

These equations describe internal forces acting on the element. Forming the load vector we should subtract components of the vector from suitable components of the global vector.