Finite Element Method
Statics of a 3D frame system

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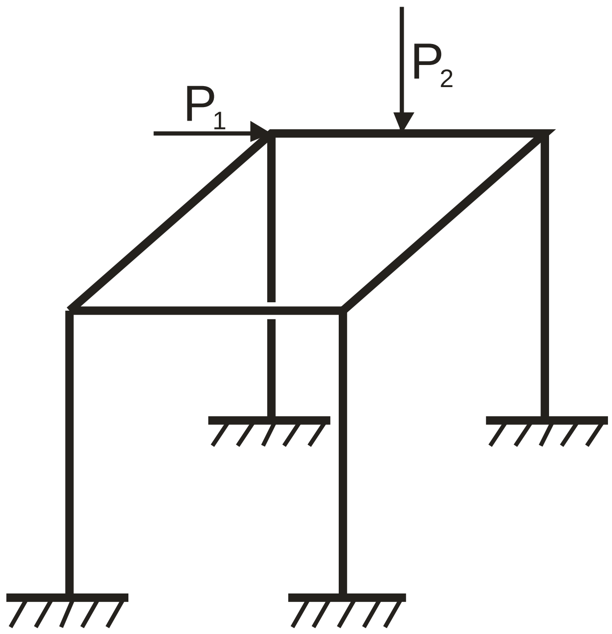
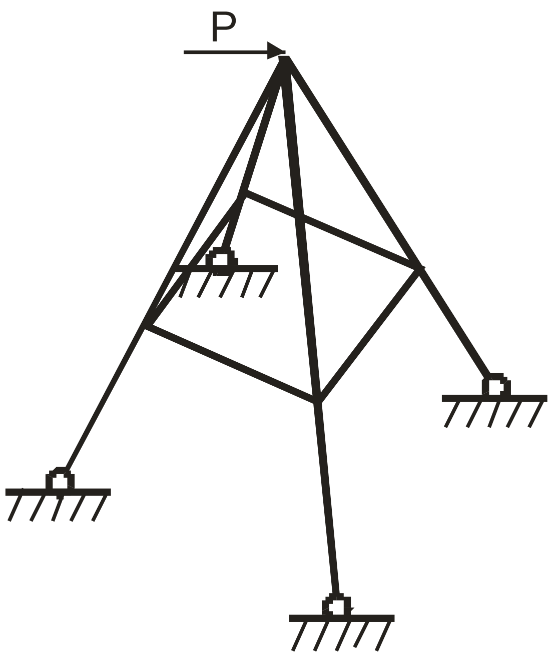
Examples of frame system

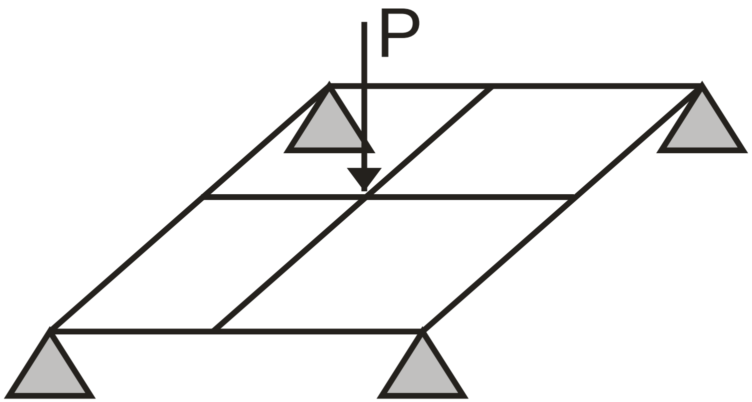
A three-dimensional frame structure is the most general type of bar structures. Elements of a space frame can serve for modelling of all the previously described structures (2D and 3D trusses, 2D frames) and some others such as grillworks, beams broken in a plane and loaded perpendicularly to its plane, etc.

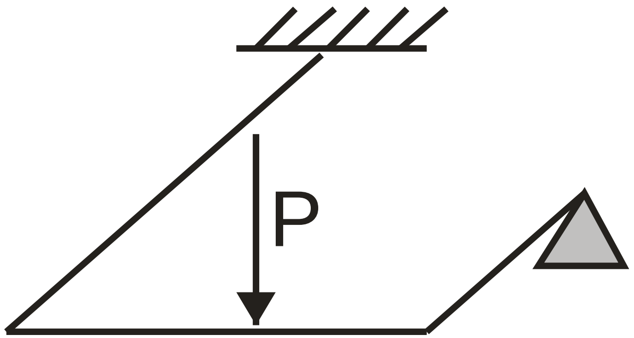
3D frame system

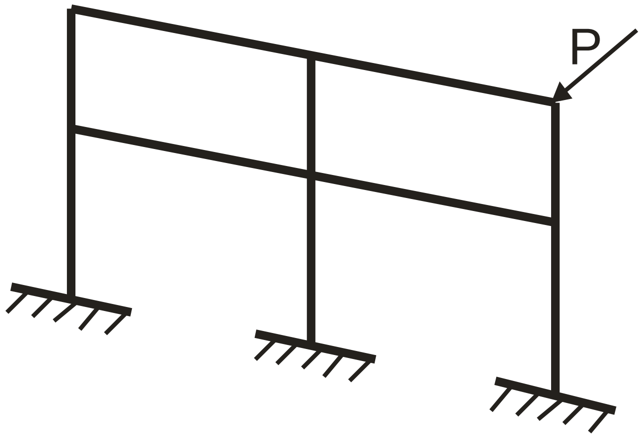




2D frame system







3D element local coordinate system

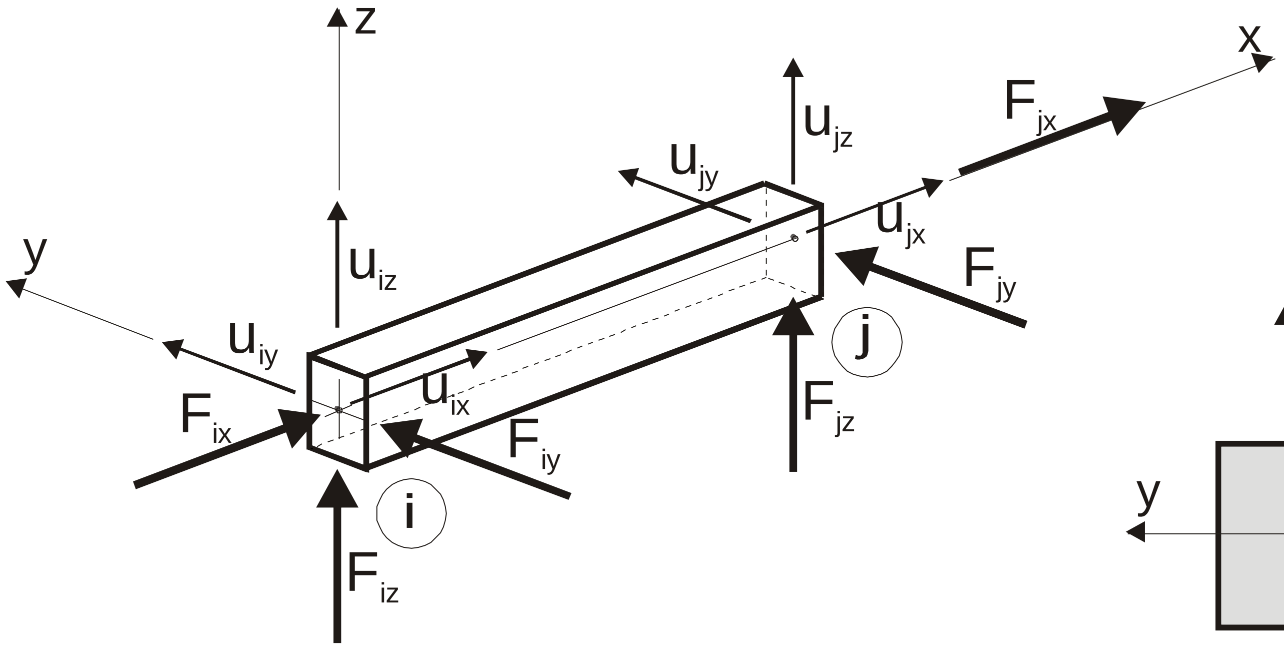
Any node of a space structure has six degrees of freedom which means that it can submit to three independent displacements and three rotations.

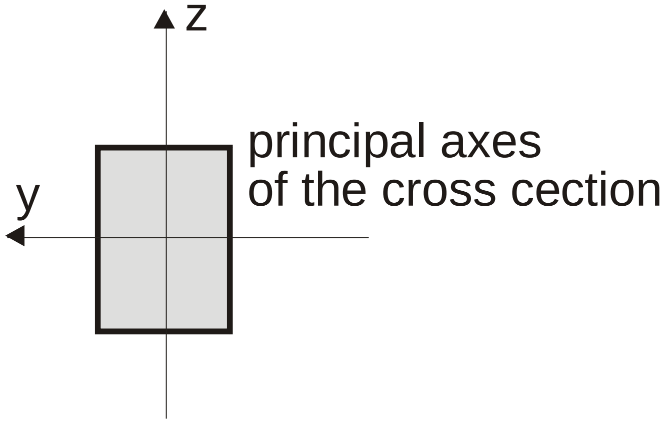
Hence a frame element has twelve degrees of freedom.

3D element local coordinate system

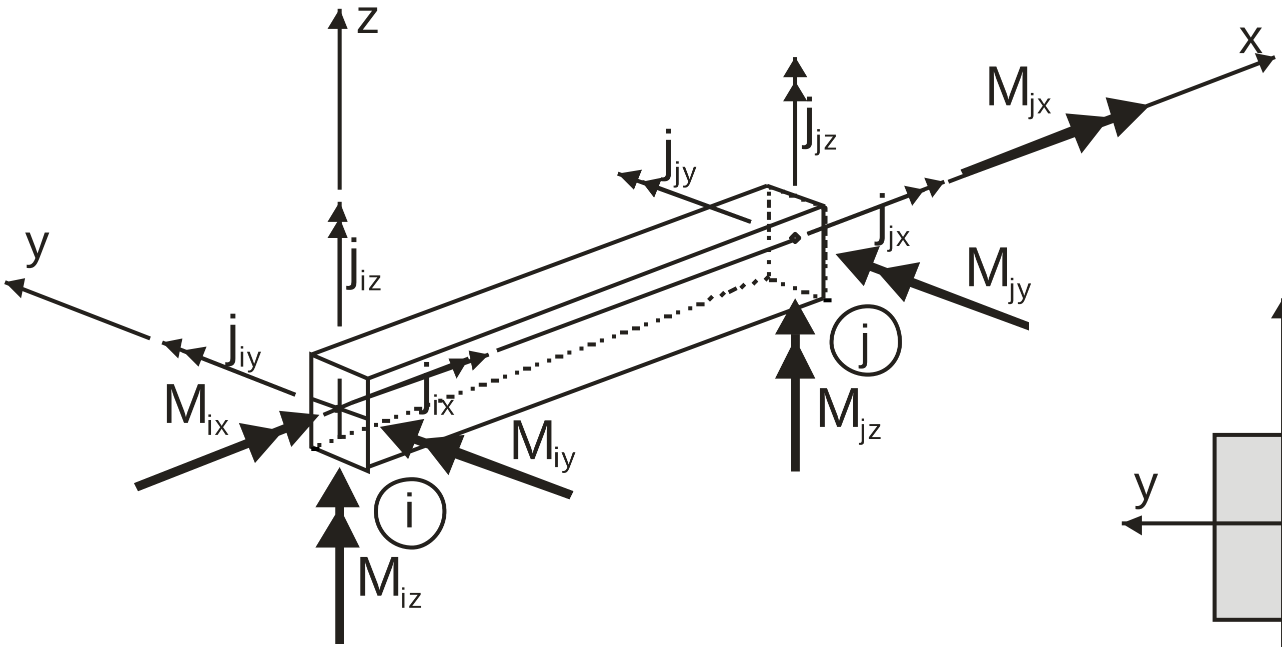
The local coordinate system has to be chosen in such a way that axes y and z are the principal axes of a cross section because it simplifies the discussion of a bending of problem. Bending of such an element can be analysed as two independent phenomena of bending in planes xy and xz.

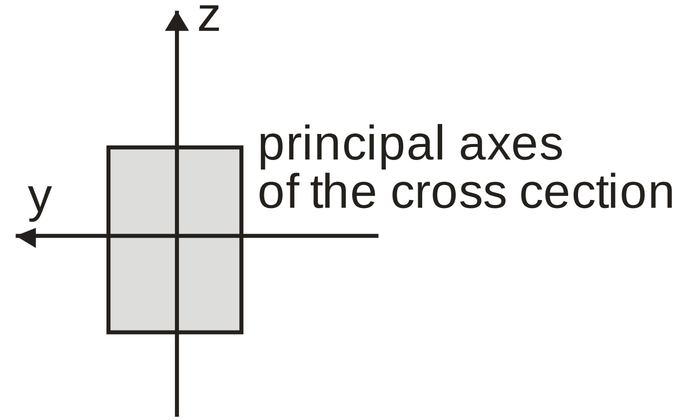
3D element local coordinate system





3D element local coordinate system





Nodal displacements vectors

$$\mathbf{u}'^{e} = \begin{bmatrix} \mathbf{u}'_i \\ \mathbf{u}'_j \end{bmatrix}$$

- nodal displacement vector of an element in the local coordinate system

Nodal displacements vectors

$$\mathbf{u}'_i = \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{iz} \\ \varphi_{ix} \\ \varphi_{iy} \\ \varphi_{iz} \end{bmatrix} \qquad \mathbf{u}'_j = \begin{bmatrix} u_{jx} \\ u_{jy} \\ u_{jz} \\ \varphi_{jx} \\ \varphi_{jy} \\ \varphi_{jz} \end{bmatrix}$$

- $\mathbf{u}'_i$ - displacement vector of the node i in the local coordinate system
- $\mathbf{u}'_j$ - displacement vector of the node j in the local coordinate system

Nodal forces vectors

$$\mathbf{f}'^{e} = \begin{bmatrix} \mathbf{f}'_i \\ \mathbf{f}'_j \end{bmatrix}$$

- nodal force vector of an element in the local system

3D element local coordinate system

$$\mathbf{f'}_i = \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \end{bmatrix} \quad \mathbf{f'}_j = \begin{bmatrix} F_{jx} \\ F_{jy} \\ F_{jz} \\ M_{jx} \\ M_{jy} \\ M_{jz} \end{bmatrix}$$

- $\mathbf{f'}_i$ - force vector of the node *i* in the local coordinate system
- $\mathbf{f'}_j$ - force vector of the node *j* in the local coordinate system

Stiffness matrix

relationship between nodal forces and displacements

$$\mathbf{f}'^{e} = \mathbf{K}'^{e}\,\mathbf{u}'^{e}$$

- $\mathbf{K}'^{e}$ - square and symmetric matrix with dimensions 12x12

Stiffness matrix

Most components of this matrix can be calculated on the basis of the results obtained for a 2D frame. Since the bending in principal planes of the cross section is independent, we will split the deformation of the element of a three-dimensional frame into a few simpler form

Deformation of the element of a 3D frame

- axial tension which is identical to that in a truss,
- bending in the *xz* plane which is similar to the states of a 2D frame; modifications concern the signs of internal forces,
- torsion.

Torsion

dependence between a nodal torsion moment and a torsion angle of an element is quite simple and resembles the relation between an axial force and an element extension:

$$\frac{\Delta\varphi_x}{L} = \frac{M_x}{GC}$$

$G = \dfrac{E}{2(1+\nu)}$ - Kirchhoff's (shear) modulus

C - torsional resistance characteristics

Torsion

$\Delta\varphi_x = \varphi_{jx} - \varphi_{ix}$ - increase the torsion angle due to the torsion moment M_x

$G = \frac{E}{2(1+\nu)}$ - Kirchhoff's modulus

C - torsional resistance characteristics

The constant C has the dimension of a moment of inertia and is equal to the polar moment of inertia for circular-symmetric sections (comp. Jastrzębski et al. (1985)) but for other sections it should be calculated by use of quite complex methods (comp. Timoshenko and Goodier (1962))

Stiffness matrix

relation between the nodal rotations around the x axis and nodal torsion moments

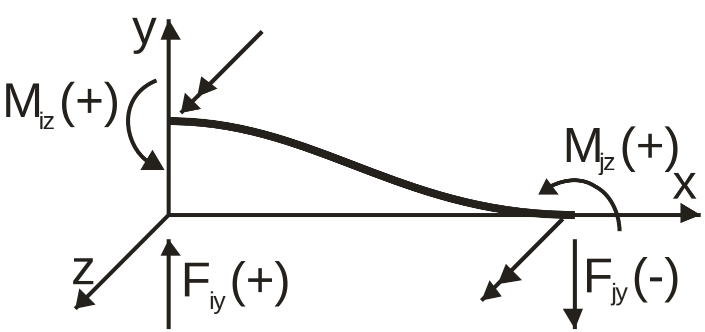
$$M_{ix} = \frac{GC}{L}\left(\varphi_{ix} - \varphi_{jx}\right)$$

$$M_{jx} = \frac{GC}{L}\left(-\varphi_{ix} + \varphi_{jx}\right)$$

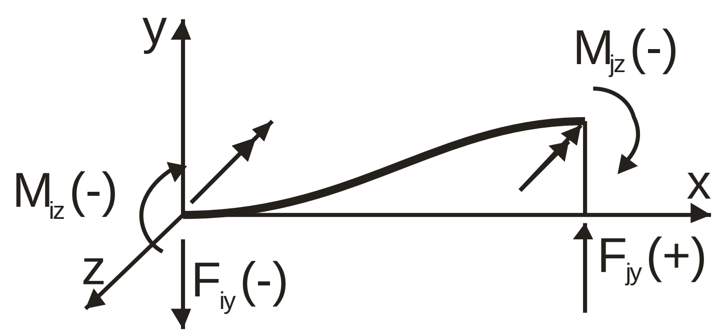
the above equations are the searched relation which allows us to write the element stiffness matrix.

Stiffness matrix

Senses of nodal forces caused by unitary nodal displacements, which allow us to determine signs of the expressions of the stiffness matrix

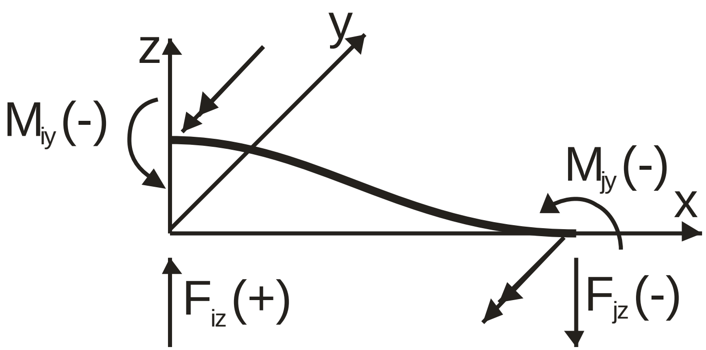


column No 2 - $u_{iy}=1$

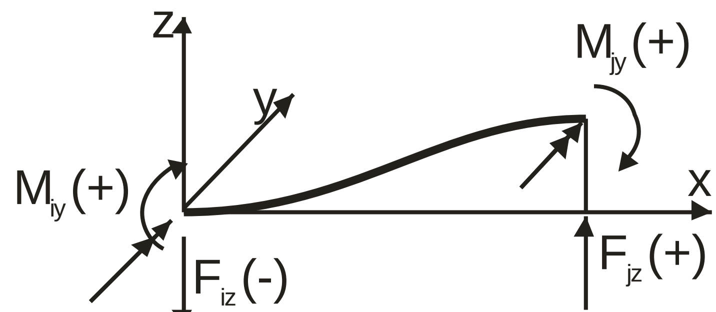


column No 8 - $u_{jy}=1$

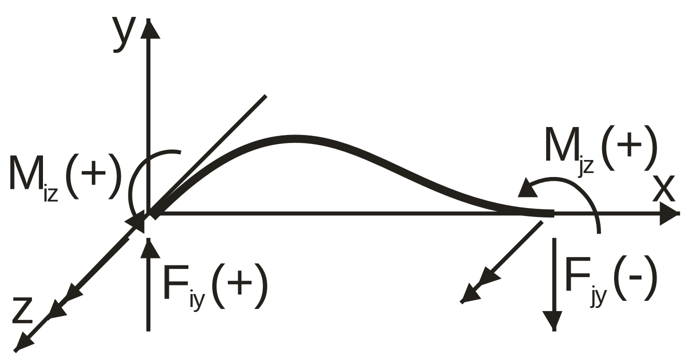
Stiffness matrix



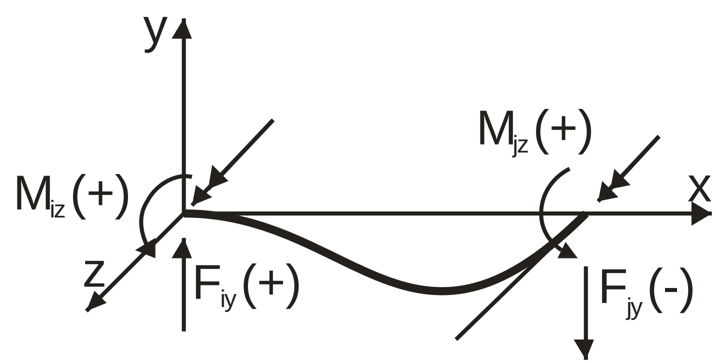
column No 3 - $u_{iz}=1$



column No 9 - $u_{jz}=1$

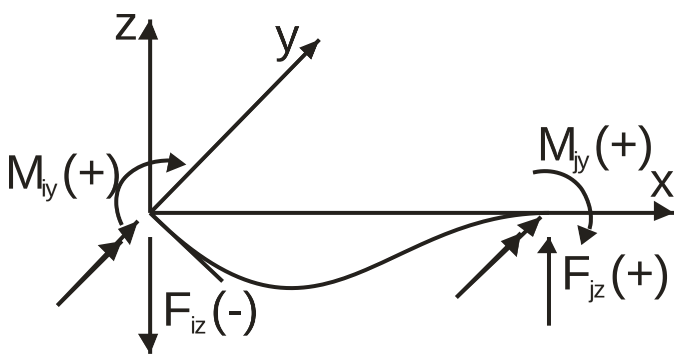


column No 6 - $j_{iz}=1$

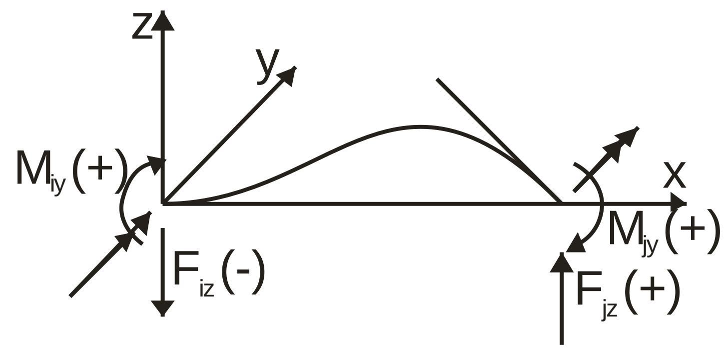


column No 12 - $j_{jz}=1$

Stiffness matrix



column No 5 - j_{iy}=1



column No 11 - j_{jy}=1

Stiffness matrix

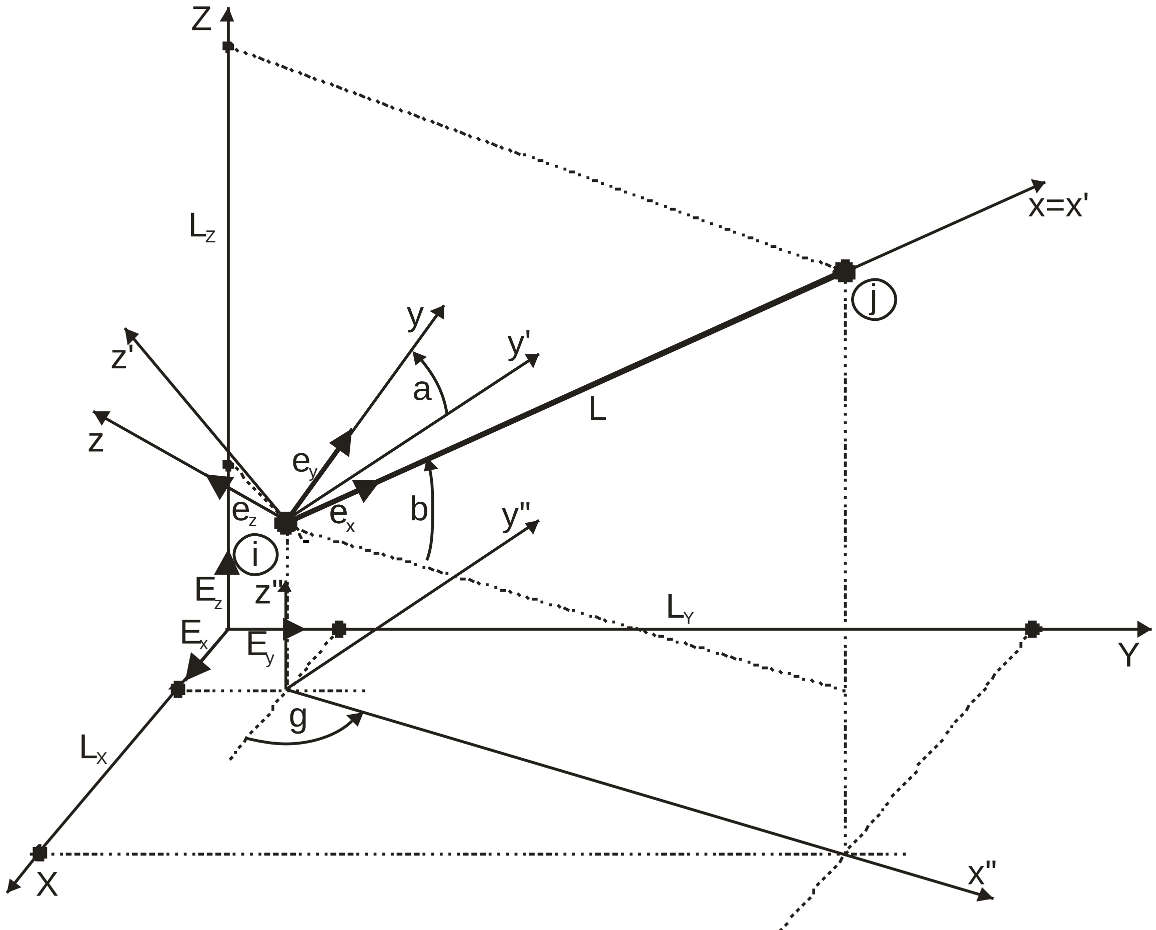
$$
\mathbf{K'}^e = \begin{array}{cc}
\begin{matrix} u_{ix} & u_{iy} & u_{iz} & \varphi_{ix} & \varphi_{iy} & \varphi_{iz} & u_{jx} & u_{jy} & u_{jz} & \varphi_{jx} & \varphi_{jy} & \varphi_{jz} \end{matrix} & \\
\left[\begin{array}{cccccc|cccccc}
\frac{EA}{L} & & & & & & -\frac{EA}{L} & & & & & \\
 & 12\frac{EJ_z}{L^3} & & & & 6\frac{EJ_z}{L^2} & & -12\frac{EJ_z}{L^3} & & & & 6\frac{EJ_z}{L^2} \\
 & & 12\frac{EJ_y}{L^3} & & -6\frac{EJ_y}{L^2} & & & & -12\frac{EJ_y}{L^3} & & -6\frac{EJ_y}{L^2} & \\
 & & & \frac{GC}{L} & & & & & & -\frac{GC}{L} & & \\
 & & -6\frac{EJ_y}{L^2} & & 4\frac{EJ_y}{L} & & & & 6\frac{EJ_y}{L^2} & & 2\frac{EJ_y}{L} & \\
 & 6\frac{EJ_z}{L^2} & & & & 4\frac{EJ_z}{L} & & -6\frac{EJ_z}{L^2} & & & & 2\frac{EJ_z}{L} \\
\hline
-\frac{EA}{L} & & & & & & \frac{EA}{L} & & & & & \\
 & -12\frac{EJ_z}{L^3} & & & & -6\frac{EJ_z}{L^2} & & 12\frac{EJ_z}{L^3} & & & & -6\frac{EJ_z}{L^2} \\
 & & -12\frac{EJ_y}{L^3} & & 6\frac{EJ_y}{L^2} & & & & 12\frac{EJ_y}{L^3} & & 6\frac{EJ_y}{L^2} & \\
 & & & -\frac{GC}{L} & & & & & & \frac{GC}{L} & & \\
 & & -6\frac{EJ_y}{L^2} & & 2\frac{EJ_y}{L} & & & & 6\frac{EJ_y}{L^2} & & 4\frac{EJ_y}{L} & \\
 & 6\frac{EJ_z}{L^2} & & & & 2\frac{EJ_z}{L} & & -6\frac{EJ_z}{L^2} & & & & 4\frac{EJ_z}{L}
\end{array}\right] &
\begin{matrix} F_{ix} \\ F_{iy} \\ F_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \\ F_{jx} \\ F_{jy} \\ F_{jz} \\ M_{jx} \\ M_{jy} \\ M_{jz} \end{matrix}
\end{array}
$$

Tranformation of the stiffness matrix

The transformation method of the matrix of a frame element is analogous to the transformation of an element of a 3D truss but the third rotation around the x axis of the local system is necessary in order to lead axes y and z to the position of the principal central axes of inertia of an element cross section. Such a choice of local axes is very important for building the stiffness matrix

Tranformation of the stiffness matrix

The location of an element in space



Building the transformation matrix

transformation of a certain displacement vector $\mathbf{u}'_i$ from the local system to the global one by the composition of three rotations:

$$\mathbf{u}_i = \mathbf{R}_\gamma \left[\mathbf{R}_\beta \left(\mathbf{R}_\alpha \mathbf{u}'_i \right) \right]$$

Building the transformation matrix

$$\mathbf{R}_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha \\ 0 & s_\alpha & c_\alpha \end{bmatrix}$$

rotation matrix around the x axis by an angle α

$$\mathbf{R}_\beta = \begin{bmatrix} c_\beta & 0 & -s_\beta \\ 0 & 1 & 0 \\ s_\beta & 0 & c_\beta \end{bmatrix}$$

rotation matrix around the y' axis by an angle β

$$\mathbf{R}_\gamma = \begin{bmatrix} c_\gamma & -s_\gamma & 0 \\ s_\gamma & c_\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

rotation matrix around the z'' axis by an angle γ

$c_\alpha = \cos\alpha$ $\quad s_\gamma = \sin\gamma$ $\quad c_\gamma = \cos\gamma$ $\quad s_\alpha = \sin\alpha$ $\quad s_\beta = \sin\beta$ $\quad c_\beta = \cos\beta$

Use of a direction vector

direction vector $\mathbf{e}_y$ which is located on the y axis of the local system and its modulus is equal to unity (such a vector is called a basic vector or a versor of an axis). Hence we have

Use of a direction vector

Hence we have:

vector of the x axis of the local system determined on the basis of element coordinates (its components are direction cosines of the element)

$$\mathbf{e}_x = \begin{bmatrix} e_{xX} \\ e_{xY} \\ e_{xZ} \end{bmatrix} = \frac{1}{L} \begin{bmatrix} L_X \\ L_Y \\ L_Z \end{bmatrix}$$

given direction vector of the element

$$\mathbf{e}_y = \begin{bmatrix} e_{yX} \\ e_{yY} \\ e_{yZ} \end{bmatrix}$$

Use of a direction vector

Since the system *xyz* is the right cartesian coordinate system, then the versors of this system are orthogonal. Thus, we can write

$$\mathbf{e}_z = \mathbf{e}_x \times \mathbf{e}_y$$

and we calculate

$$\mathbf{e}_z = \begin{bmatrix} e_{zX} \\ e_{zY} \\ e_{zZ} \end{bmatrix} \qquad e_{zX} = \begin{vmatrix} e_{xY} & e_{xZ} \\ e_{yY} & e_{yZ} \end{vmatrix} \qquad e_{zZ} = \begin{vmatrix} e_{xX} & e_{xY} \\ e_{yX} & e_{yY} \end{vmatrix} \qquad e_{zY} = -\begin{vmatrix} e_{xX} & e_{xZ} \\ e_{yX} & e_{yZ} \end{vmatrix}$$

Use of a direction vector

Since any vector can be presented as a sum of products of its coordinates and versors, then we obtain:

$$\mathbf{u} = u_x \mathbf{e}_x + u_y \mathbf{e}_y + u_z \mathbf{e}_z =$$

$$u_x \left(e_{xX} \mathbf{E}_X + e_{xY} \mathbf{E}_Y + e_{xZ} \mathbf{E}_Z \right) + u_y \left(e_{yX} \mathbf{E}_X + e_{yY} \mathbf{E}_Y + e_{yZ} \mathbf{E}_Z \right) + u_z \left(e_{zX} \mathbf{E}_X + e_{zY} \mathbf{E}_Y + e_{zZ} \mathbf{E}_Z \right) =$$

$$\left(u_x e_{xX} + u_y e_{yX} + u_z e_{zX} \right) \mathbf{E}_X + \left(u_x e_{xY} + u_y e_{yY} + u_z e_{zY} \right) \mathbf{E}_Y + \left(u_x e_{xZ} + u_y e_{yZ} + u_z e_{zZ} \right) \mathbf{E}_Z$$

or less

$\mathbf{u}_i = \mathbf{R}_i \mathbf{u}'_i$ where $\mathbf{R}_i$ is the rotation matrix of a node

Use of a direction point

Here we present one of the possibilities of simplifying the way of passing the direction of an element axis which is used in the *Autodesk Simulation Mechanical (Algor)* system. The 3D frame element is determined by three points (i - the first node, j - the last node, k - the direction node).

Use of a direction point

The points i, j, k determine a plane in the three dimensional space. The axis y of the local coordinate system is in this plane. The x axis is determined by the line passing through points i, j. We find coordinates of *versors* for such a definition of directions of the local axes.

Use of a direction point

X_i, Y_i, Z_i denote coordinates of the point i in the global system. If analogy, we denote coordinates of points j and k, then the element coordinates in the global system are equal to:

$$L_X = X_j - X_i, \quad L_Y = Y_j - Y_i, \quad L_Z = Z_j - Z_i, \quad L = \sqrt{L_X^2 + L_Y^2 + L_Z^2}$$

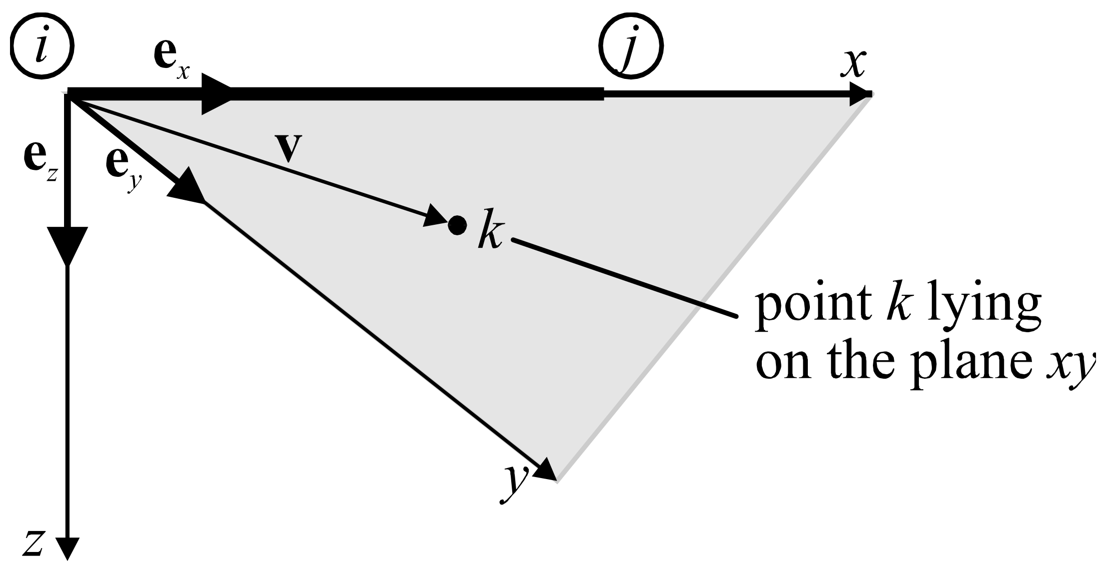
and from here we calculate the components of vector $\mathbf{e}_x$

$$e_{xX} = \frac{L_X}{L} \qquad e_{xY} = \frac{L_Y}{L} \qquad e_{xZ} = \frac{L_Z}{L}$$

Use of a direction point

We form the vector **v** connecting the point i and the direction point k

$$\mathbf{v} = \begin{bmatrix} X_k - X_i \\ Y_k - Y_i \\ Z_k - Z_i \end{bmatrix}$$



Use of a direction point

The vector product of the vectors $\mathbf{e}_x$ and $\mathbf{v}$ give a vector which is perpendicular to the xy plane. This vector will be the versor $\mathbf{e}_z$

$$\mathbf{w} = \mathbf{e}_x \times \mathbf{v}$$

$$w_X = \begin{vmatrix} e_{xY} & e_{xZ} \\ v_Y & v_Z \end{vmatrix} \qquad w_Y = -\begin{vmatrix} e_{xX} & e_{xZ} \\ v_X & v_Z \end{vmatrix} \qquad w_Z = \begin{vmatrix} e_{xX} & e_{xY} \\ v_X & v_Y \end{vmatrix}$$

$$w = \sqrt{w_X^2 + w_Y^2 + w_Z^2}$$

$$\mathbf{e}_{zX} = \frac{w_X}{w} \qquad \mathbf{e}_{zY} = \frac{w_Y}{w} \qquad \mathbf{e}_{zZ} = \frac{w_Z}{w}$$

The transformation matrix of an element

Nodal displacement vectors and nodal force vectors have been grouped so that we can divide them into blocks containing either displacements or rotations and either forces or moments respectively. After this operation we can transform every block independently

The transformation matrix of an element

$$\mathbf{R}^e = \begin{bmatrix} \mathbf{R}_i & & & \\ & \mathbf{R}_i & & \\ & & \mathbf{R}_j & \\ & & & \mathbf{R}_j \end{bmatrix}$$

$\mathbf{R}_i$ - rotation matrix of the node i

$\mathbf{R}_j$ - rotation matrix of the node j

We obtain the transformation of the stiffness matrix to the global system by multiplying matrices

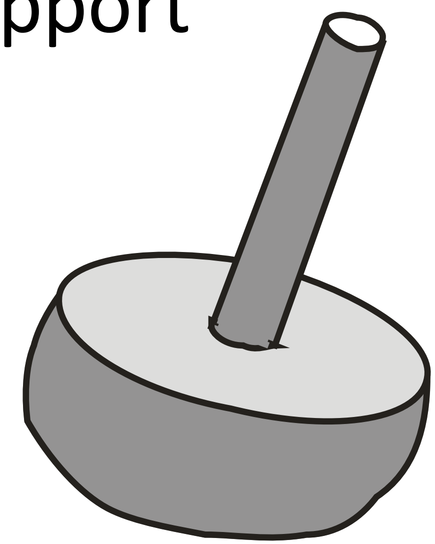
$$\mathbf{K}^e = \mathbf{R}^e \mathbf{K}'^e \left(\mathbf{R}^e\right)^\mathsf{T}$$

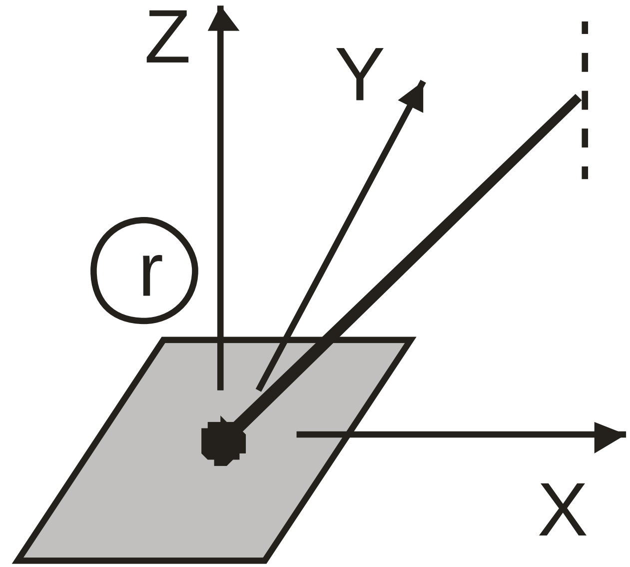
Boundary conditions for a 3D frame

Boundary conditions existing in 3D frame supports are very similar to conditions described for two-dimensional frames. Differences concerning degrees of freedom which do not exist in plane frames are obvious. We elaborate only those boundary conditions which describe frame supports of space structures and which are most often applied

3D frame support types

rigid support





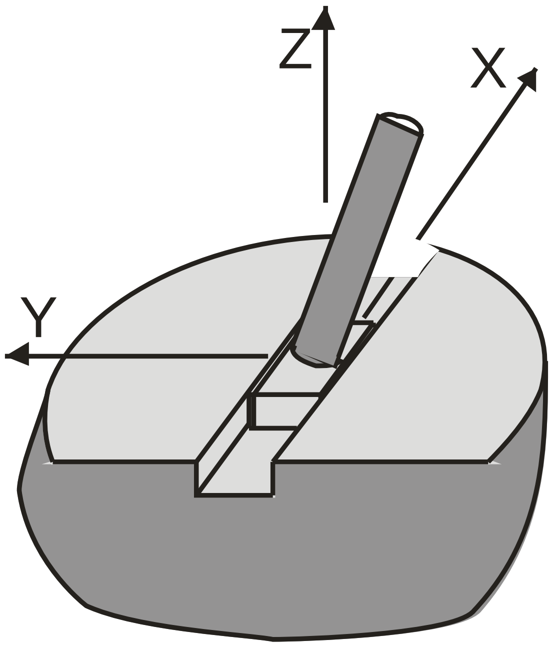
$u_{rX} = 0 \quad j_{rX} = 0$

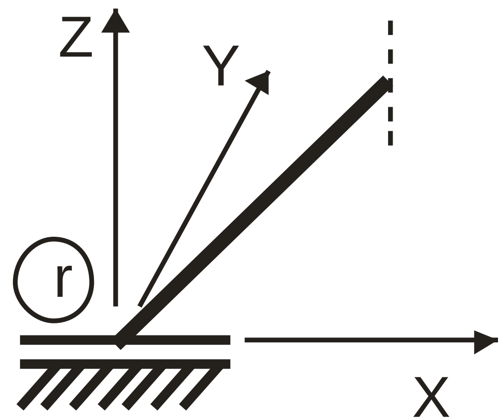
$u_{rY} = 0 \quad j_{rY} = 0$

$u_{rZ} = 0 \quad j_{rZ} = 0$

3D frame support types

Linear moveable support (along the X axis)





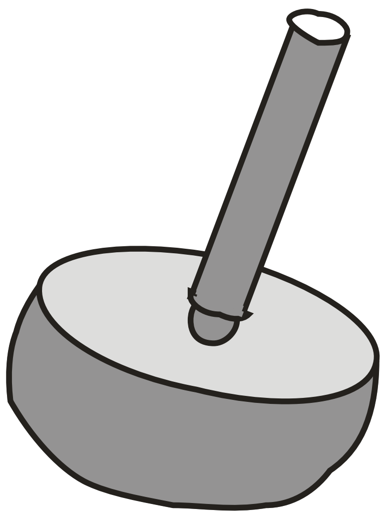
$u_{rY} = 0 \quad j_{rX} = 0$

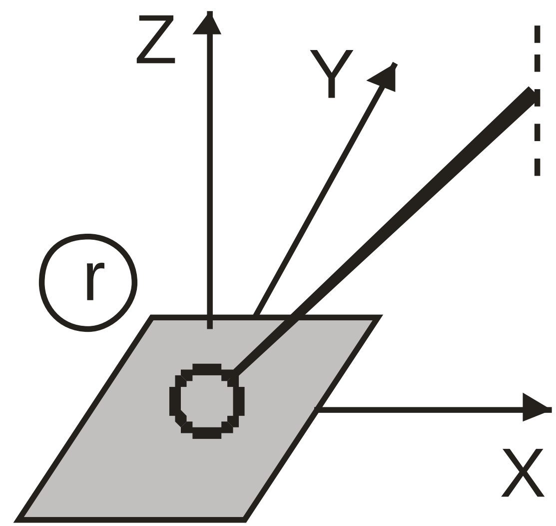
$u_{rZ} = 0 \quad j_{rY} = 0$

$j_{rZ} = 0$

3D frame support types

ball-shaped joint

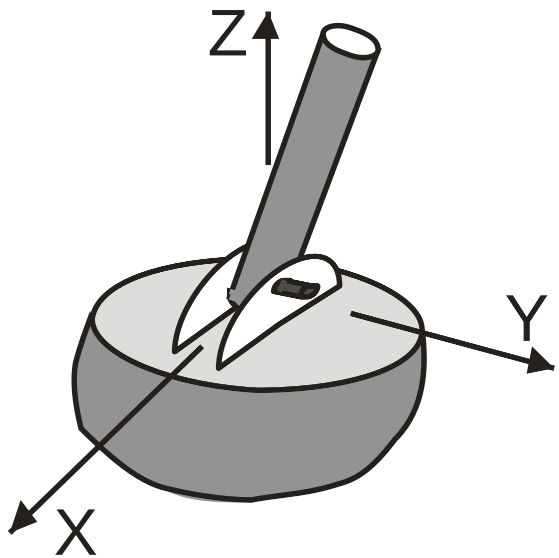


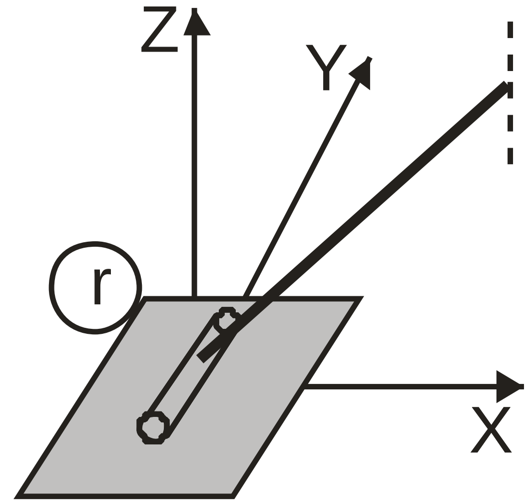


$$u_{rX} = 0$$
$$u_{rY} = 0$$
$$u_{rZ} = 0$$

3D frame support types

cylindrical joint





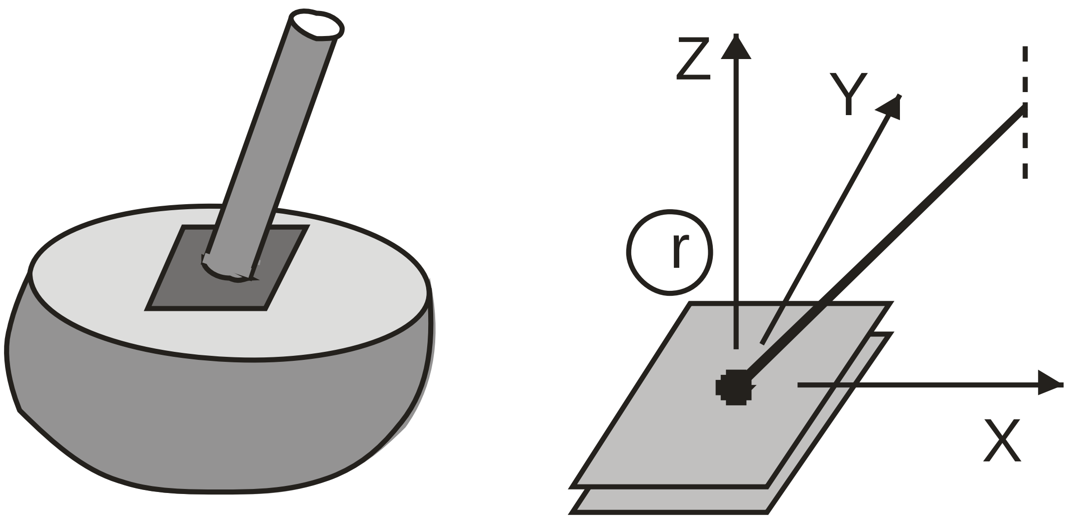
$u_{rX} = 0 \quad j_{rX} = 0$

$u_{rY} = 0 \quad j_{rZ} = 0$

$u_{rZ} = 0$

3D frame support types

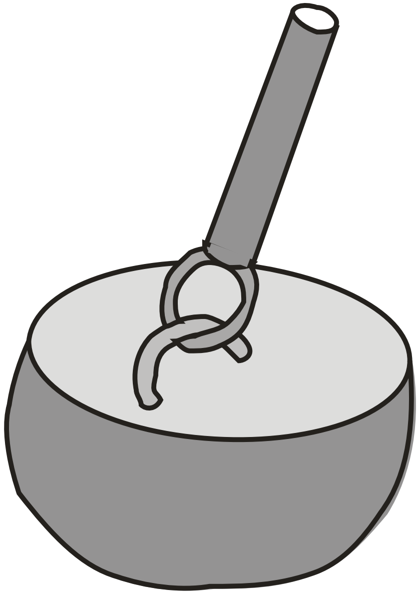
moveable plane support



$$u_{rZ} = 0$$

3D frame support types

cardan joint



$u_{rX} = 0$

$u_{rY} = 0$

$u_{rZ} = 0 \quad j_{rZ} = 0$

Boundary elements

As in previous, we propose to use elastic and fixed boundary elements for modelling these constraints. In fact we can use a single element of which we can compose a more complex support but for convenience we will show here the use of the matrix of a versatile elastic element with six degrees of freedom

Boundary elements

$$\mathbf{K}'^{b} = \begin{bmatrix} h_{rX} & 0 & 0 & 0 & 0 & 0 \\ 0 & h_{rY} & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{rZ} & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{rX} & 0 & 0 \\ 0 & 0 & 0 & 0 & g_{rY} & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{rZ} \end{bmatrix}$$

- h_{rX}, h_{rY}, h_{rZ} - spring rates
- g_{rX}, g_{rY}, g_{rZ} - flexural (or torsion) stiffness of springs

Boundary elements

transformation of matrix to the global system

$$\mathbf{R}^b = \begin{bmatrix} \mathbf{R}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_r \end{bmatrix}$$

- $\mathbf{R}_r$ - rotation matrix of the node

After the multiplication we obtain the stiffness matrix of the boundary element in the global coordinate system

$$\mathbf{K}^b = \mathbf{R}^b \mathbf{K}'^b \left(\mathbf{R}^b\right)^{\mathrm{T}} = \begin{bmatrix} \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} \end{bmatrix}$$

- $\mathbf{H}$ - stiffness matrix for a movement
- $\mathbf{G}$ - stiffness matrix for a rotation

Boundary elements

It is easy to obtain the matrix $\mathbf{G}$ from the matrix $\mathbf{H}$ changing the stiffness of tension of springs h_{rX}, h_{rY}, h_{rZ} into the stiffness of bending springs g_{rX}, g_{rY}, g_{rZ}

$$\mathbf{H} = \begin{bmatrix} e_{xX}^2 h_{rX} + e_{yX}^2 h_{rY} + e_{zX}^2 h_{rZ} & 0 & 0 \\ 0 & e_{xY}^2 h_{rX} + e_{yY}^2 h_{rY} + e_{zY}^2 h_{rZ} & 0 \\ 0 & 0 & e_{xZ}^2 h_{rX} + e_{yZ}^2 h_{rY} + e_{zZ}^2 h_{rZ} \end{bmatrix}$$

$$\boldsymbol{G} = \begin{bmatrix} e_{xX}^2 g_{rX} + e_{yX}^2 g_{rY} + e_{zX}^2 g_{rZ} & 0 & 0 \\ 0 & e_{xY}^2 g_{rX} + e_{yY}^2 g_{rY} + e_{zY}^2 g_{rZ} & 0 \\ 0 & 0 & e_{xZ}^2 g_{rX} + e_{yZ}^2 g_{rY} + e_{zZ}^2 g_{rZ} \end{bmatrix}$$