Finite Element Method
Two-dimensional elements

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Introduction

Structures discussed in the previous chapters were modelled by means of bar structures whose equilibrium equations as well as their geometrical relationships are described with the help of differential equilibrium equations and whose independent variable is measured along the bar axis.

Introduction

This rather simple structure lets us get familiar with the essence of the FEM and convinces the reader that this method is efficient in solving very complex and extended problems in structural mechanics. Now, we will discuss surface structures such as 2D elements, plate and shell for which displacements, strains, internal forces are the functions of two independent coordinates

Introduction

Differential equilibrium equations for bar structures are simple enough to be integrated. Their exact results can be used as element shape functions. The situation is quite different for surface structures. Partial differential equations describing the equilibrium of those structures have unique solutions only for very simple problems

Introduction

Solutions obtained by using the approximation method (for example, by expansion in a series) are very laborious and they require a lot of work and therefore a computer has to be used in order to solve a set of equations and sum series

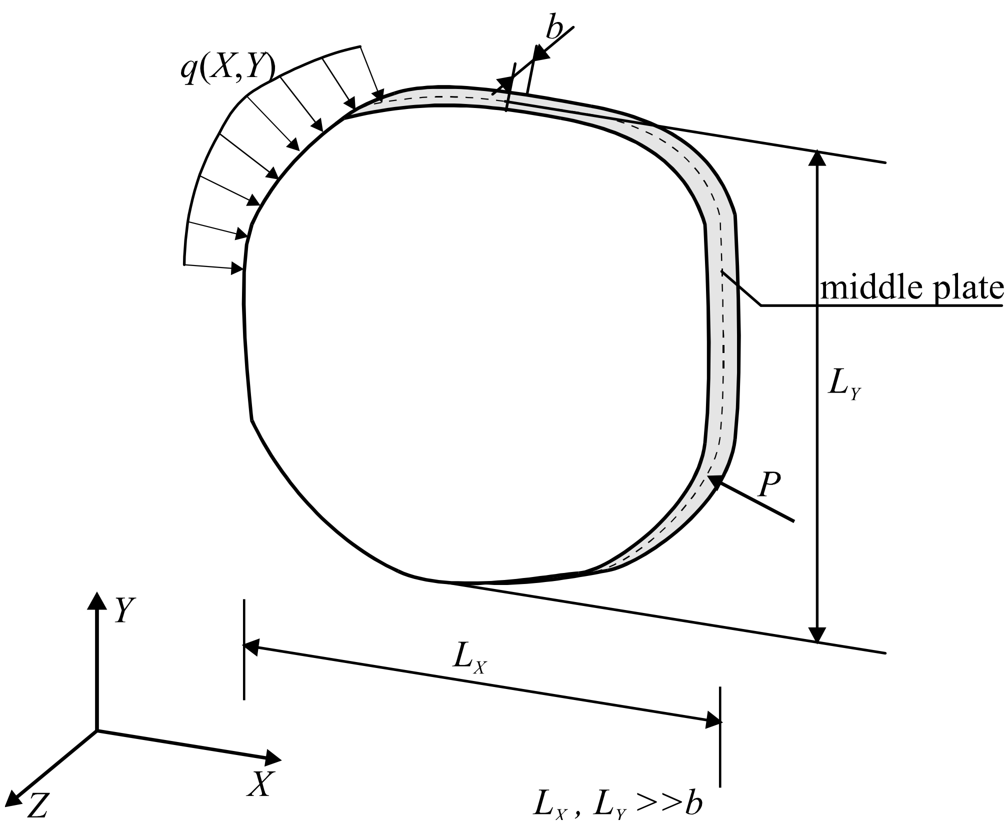
Introduction

In such a situation, a numerical method which assumes some simplification at the stage of formation of element equilibrium equations appears to be more effective. That is why the finite element method has brought so many significant results to continuum mechanics

Introduction

The 2D element can be defined as a solid of which one dimension (thickness) is considerably smaller than the two others and whose middle plane (the surface parallel to both external surfaces of an element) is a plane

Introduction



Introduction

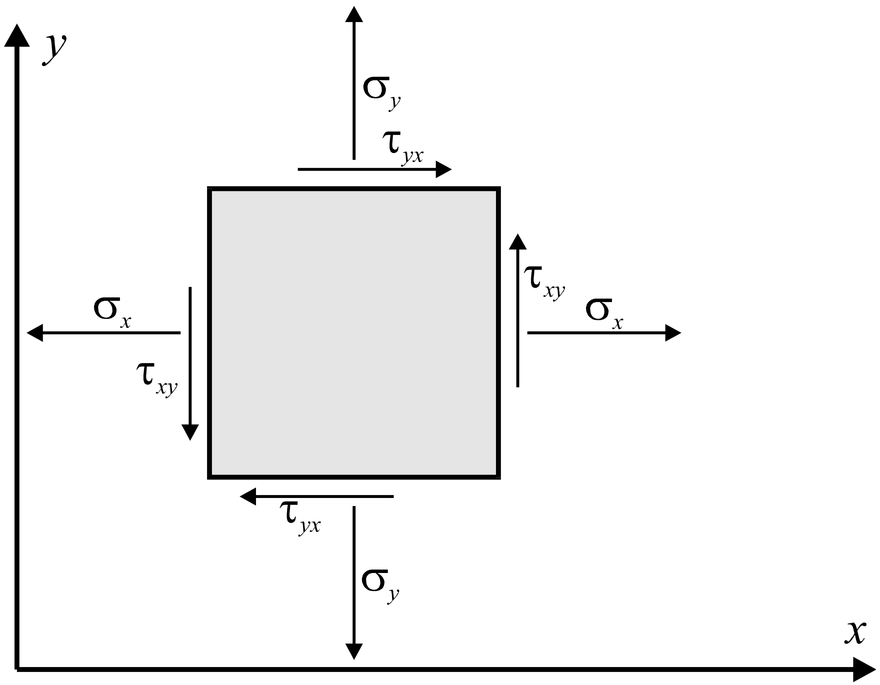
A plate element has also such a shape but the 2D element differs from a plate the way it is loaded. The 2D element can be loaded only with the load acting in its plane and by the temperature dependent upon the x and y coordinates. On the other hand, the plate can be loaded with a force perpendicular to its surface or any temperature field

Plane stress and strain

When external surfaces of a 2D element are free and this element is thin enough, we can assume that in reference to the whole thickness of the element. Then it is said that this is a plane stress problem

Plane stress and strain

Hence only the components of stress shown in Figure are non-zero



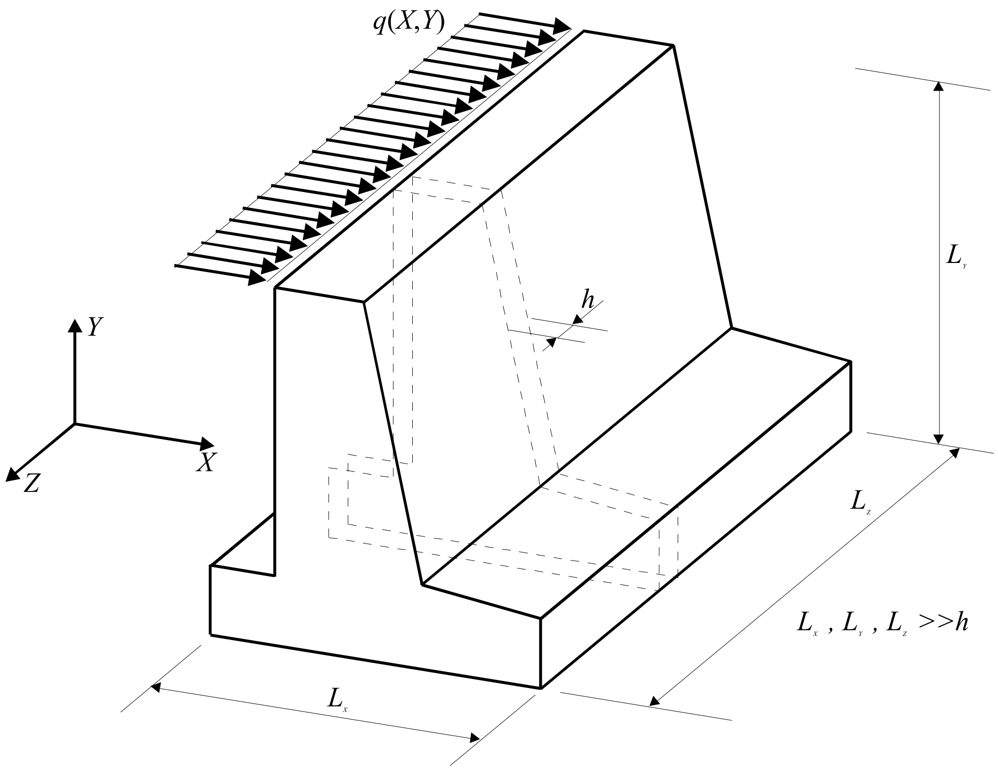
Plane stress and strain

With regard to the symmetry of a stress tensor components of shear stress and are equal, thus we have three independent components of stress which we compose in the stress vector

$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

Plane stress and strain

A completely different case occurs when the component LZ in Figure is very significant



Plane stress and strain

when $h << LX, LY, LZ$, and the support and load conditions are constant along the axis which is perpendicular to the element. The structure satisfying these conditions can also be analysed by applying plane state which in fact is plane strain. Since the cross dimension of the structure prevents the structure deformation in the direction perpendicular to the cross section

Plane stress and strain

thin layer cut out from this structure is in the state described by the equation

$$\varepsilon_z = 0, \gamma_{zx} = 0, \gamma_{zy} = 0$$

$\sigma_z \neq 0$ comes from the above equations, but the first equation allows to calculate the component on the basis of two other components of a direct stress. Thus, we have

$$\sigma_z = \nu\left(\sigma_x + \sigma_y\right)$$

Plane stress and strain

We also group independent components of the strain tensor in a column matrix which we have called a strain vector

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

Plane stress and strain

There is a relationship between vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ described by constitutive equations whose form depends on the model of the material which the structure is made of. We deal only with elastic isotropic materials which obey Hook's law. Hence we can write the constitutive equation as follows

$$\boldsymbol{\sigma} = \mathbf{D} \cdot \boldsymbol{\varepsilon}$$

Geometric relationships

A certain point can move only on the plane during the deformation process and then the displacement vector of this point u(x,y) has two components

$$\mathbf{u}(x,y) = \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix}$$

Geometric relationships

Some known relations exist between the components of displacement and strain vectors

$$\varepsilon_x = \frac{\partial u_x}{\partial x} \qquad \varepsilon_y = \frac{\partial u_y}{\partial y} \qquad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

Geometric relationships

which can be presented in the form

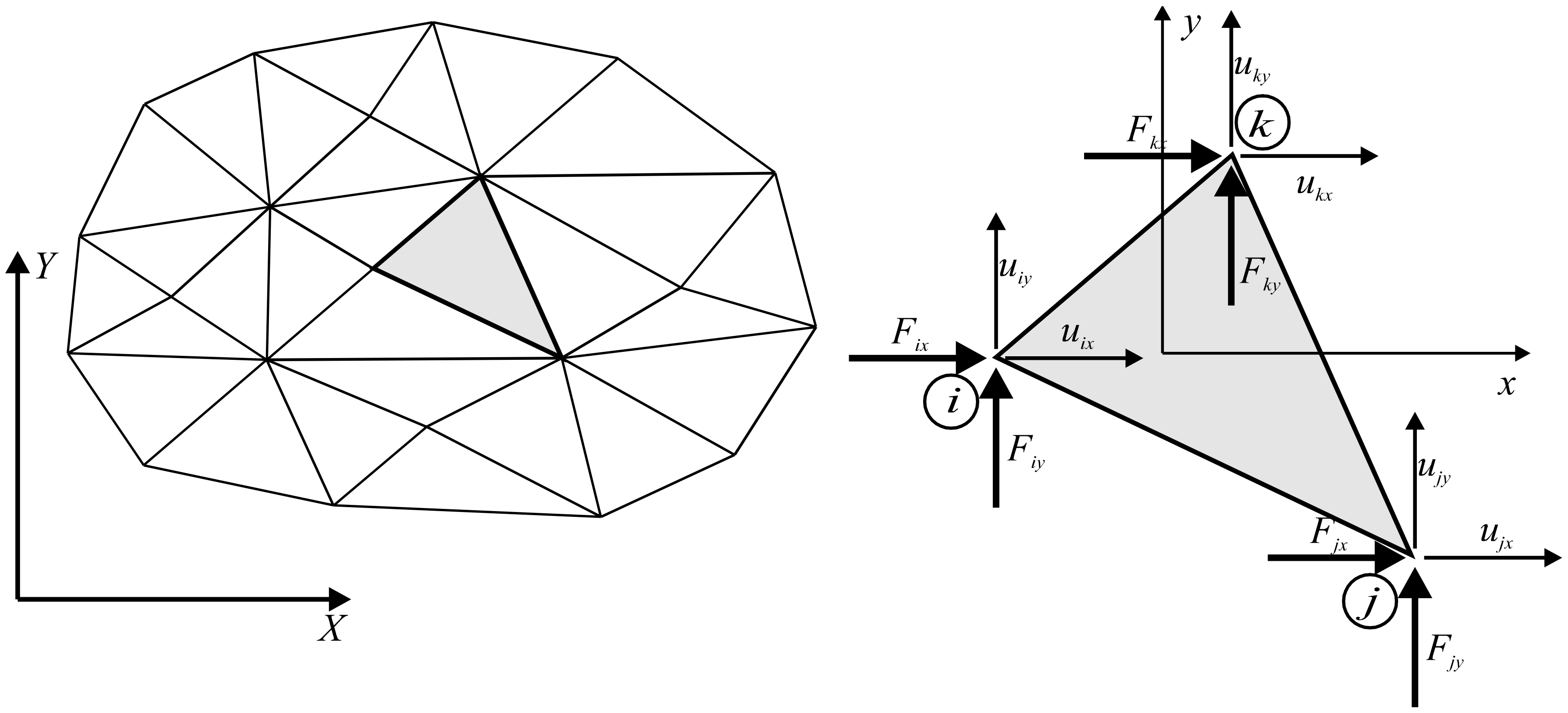
$$\boldsymbol{\varepsilon} = \mathbf{D} \cdot \mathbf{u}(x, y)$$

$\mathbf{D}$ is the matrix of differential operators Eqn.

The stiffness matrix of an elastic element

Let us divide a continuum into finite elements. We will discuss only a triangular 2D element and we will choose such elements during discretization

The stiffness matrix of an elastic element



The stiffness matrix of an elastic element

every node of an element has two degrees of freedom and all nodal forces have two components. The local coordinate system xy is chosen in such a way that its axes are parallel to the axes of the global coordinate system

The stiffness matrix of an elastic element

nodal and element displacements

.

$$\mathbf{u}_i = \begin{bmatrix} u_{ix} \\ u_{iy} \end{bmatrix} \qquad \mathbf{u}_j = \begin{bmatrix} u_{jx} \\ u_{jy} \end{bmatrix} \qquad \mathbf{u}_k = \begin{bmatrix} u_{kx} \\ u_{ky} \end{bmatrix} \qquad \mathbf{u}^e = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \\ \mathbf{u}_k \end{bmatrix} = \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{jx} \\ u_{jy} \\ u_{kx} \\ u_{ky} \end{bmatrix}$$

nodal and element forces

$$\mathbf{f}_i = \begin{bmatrix} F_{ix} \\ F_{iy} \end{bmatrix} \qquad \mathbf{f}_j = \begin{bmatrix} F_{jx} \\ F_{jy} \end{bmatrix} \qquad \mathbf{f}_k = \begin{bmatrix} F_{kx} \\ F_{ky} \end{bmatrix} \qquad \mathbf{f}^e = \begin{bmatrix} \mathbf{f}_i \\ \mathbf{f}_j \\ \mathbf{f}_k \end{bmatrix} = \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{jx} \\ F_{jy} \\ F_{kx} \\ F_{ky} \end{bmatrix}$$

The stiffness matrix of an elastic element

Since we look for the dependence between nodal displacement and nodal forces vectors of an element we apply the principle of virtual work which requires giving the relation between displacements of points lying within the element and displacements of nodes

The stiffness matrix of an elastic element

Accepting errors coming from approximation, we assume that this relationship can be written by the function of two variables

$$u_x(x,y) = N_i(x,y)u_{ix} + N_j(x,y)u_{jx} + N_k(x,y)u_{kx}$$

$$u_y(x,y) = N_i(x,y)u_{iy} + N_j(x,y)u_{jy} + N_k(x,y)u_{ky}$$

or the general matrix form

$$\mathbf{u}(x,y) = \mathbf{N}^e(x,y)\,\mathbf{u}^e$$

The stiffness matrix of an elastic element

$\mathbf{N}^e(x,y)$ is the matrix of shape functions of the element

$$\mathbf{N}^e(x,y) = \left[N_i(x,y)\,\mathbf{I} \quad N_j(x,y)\,\mathbf{I} \quad N_k(x,y)\,\mathbf{I} \right]$$

$N_i\,(x,y)$, $N_j\,(x,y)$, $N_k\,(x,y)$ are the shape functions for nodes i, j, k

The stiffness matrix of an elastic element

Let us now assume the simplest of all possible forms of the shape function for the node i

$$N_i(x, y) = a_i + b_i x + c_i y$$

a_i, b_i, c_i are constants which we determine on the basis of consistency conditions

$$N_i(x_i, y_i) = 1 \qquad N_i(x_j, y_j) = 0 \qquad N_i(x_k, y_k) = 0$$

The stiffness matrix of an elastic element

$$\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

after solving this set of equations, we get the values of coefficients of the shape function

in general form $\mathbf{M}\boldsymbol{\alpha}_i = \boldsymbol{\delta}_i, \text{where } \boldsymbol{\delta}_i = \begin{bmatrix} \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \end{bmatrix}$

The stiffness matrix of an elastic element

general form, after modification depending on the change of i into j (or k), allows us to determine the coefficients of the shape functions for the subsequent nodes. δ_{ij} means the Kronecker's delta in this equation

The stiffness matrix of an elastic element

We solve the set of equation by the Cramer method

$$W = \det \mathbf{M} = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} = \begin{vmatrix} x_j & y_j \\ x_k & y_k \end{vmatrix} - \begin{vmatrix} x_i & y_i \\ x_k & y_k \end{vmatrix} + \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix}$$

The stiffness matrix of an elastic element

$$W_{a_i} = \begin{vmatrix} 1 & x_i & y_i \\ 0 & x_j & y_j \\ 0 & x_k & y_k \end{vmatrix} = \begin{vmatrix} x_j & y_j \\ x_k & y_k \end{vmatrix}$$

$$W_{b_i} = \begin{vmatrix} 1 & 1 & y_i \\ 1 & 0 & y_j \\ 1 & 0 & y_k \end{vmatrix} = -\begin{vmatrix} 1 & y_i \\ 1 & y_k \end{vmatrix} = y_j - y_k$$

$$W_{c_i} = \begin{vmatrix} 1 & x_i & 1 \\ 1 & x_j & 0 \\ 1 & x_k & 0 \end{vmatrix} = \begin{vmatrix} 1 & y_j \\ 1 & y_k \end{vmatrix} = x_k - x_j$$

then $a_i = \dfrac{W_{a_i}}{W}$ $\quad b_i = \dfrac{W_{b_i}}{W}$ $\quad c_i = \dfrac{W_{c_i}}{W}$

The stiffness matrix of an elastic element

Similarly, if we change the index *i* into *j* and we find

$$\boldsymbol{\delta}_j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$W_{a_j} = \begin{vmatrix} 0 & x_i & y_i \\ 1 & x_j & y_j \\ 0 & x_k & y_k \end{vmatrix} = -\begin{vmatrix} x_i & y_i \\ x_k & y_k \end{vmatrix} \qquad W_{b_j} = \begin{vmatrix} 1 & 0 & y_i \\ 1 & 1 & y_j \\ 1 & 0 & y_k \end{vmatrix} = y_k - y_i$$

$$W_{c_j} = \begin{vmatrix} 1 & x_i & 0 \\ 1 & x_j & 1 \\ 1 & x_k & 0 \end{vmatrix} = x_i - x_k \qquad a_j = \frac{W_{a_j}}{W} \qquad b_j = \frac{W_{b_j}}{W} \qquad c_j = \frac{W_{c_j}}{W}$$

The stiffness matrix of an elastic element

Finally, for node k we have

$$\boldsymbol{\delta}_k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$W_{a_k} = \begin{vmatrix} 0 & x_i & y_i \\ 0 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} = \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix} \qquad W_{b_k} = \begin{vmatrix} 1 & 0 & y_i \\ 1 & 0 & y_j \\ 1 & 1 & y_k \end{vmatrix} = y_i - y_j$$

$$W_{c_k} = \begin{vmatrix} 1 & x_i & 0 \\ 1 & x_j & 0 \\ 1 & x_k & 1 \end{vmatrix} = x_j - x_i \qquad a_k = \frac{W_{a_k}}{W} \quad b_k = \frac{W_{b_k}}{W} \quad c_k = \frac{W_{c_k}}{W}$$

The stiffness matrix of an elastic element

After determining the shape functions of the element, let us come back to its strains

$$\boldsymbol{\varepsilon} = \mathbf{D}\ \mathbf{N}^e(x,y)\mathbf{u}^e = \mathbf{B}^e(x,y)\mathbf{u}^e$$

The stiffness matrix of an elastic element

The matrix $\mathbf{B}$ is called a geometric matrix and it can be expressed as follows

$$\mathbf{B}^e(x,y) = \begin{bmatrix} \mathbf{B}_i(x,y) & \mathbf{B}_j(x,y) & \mathbf{B}_k(x,y) \end{bmatrix}$$

where $\mathbf{B}_n = \mathbf{D}\ \mathbf{N}_n(x,y) = \begin{bmatrix} b_n & 0 \\ 0 & c_n \\ c_n & b_n \end{bmatrix}$ is the geometric matrix of any node n

The stiffness matrix of an elastic element

Thus, we have all components which are necessary to write an element equilibrium equation. We apply the principle of virtual work which says that the external work (done by external forces - here nodal forces) has to be equal to internal work (done by stress) of a 2D element

$$\left(\mathbf{u}^e\right)^{\mathsf{T}} \mathbf{f}^e = \int_{\mathcal{V}} \varepsilon^{\mathsf{T}} \sigma d\mathcal{V}$$

The stiffness matrix of an elastic element

$$\left(\mathbf{u}^e\right)^{\mathrm{T}} \mathbf{f}^e = \int_{\mathcal{V}} \left(\mathbf{B}^e \mathbf{u}^e\right)^{\mathrm{T}} \mathbf{D} \mathbf{B}^e \mathbf{u}^e d\mathcal{V} = \left(\mathbf{u}^e\right)^{\mathrm{T}} \int_{\mathcal{V}} \left(\mathbf{B}^e\right)^{\mathrm{T}} \mathbf{D} \mathbf{B}^e d\mathcal{V} \, \mathbf{u}^e$$

In this equation the nodal displacement vectors of the element being independent of variables x and y, are taken to the front and back of the integral. Equation can be solved independently of element displacements only when

$$\mathbf{f}^e = \int_{\mathcal{V}} \left(\mathbf{B}^e\right)^{\mathrm{T}} \mathbf{D} \mathbf{B}^e d\mathcal{V} \, \mathbf{u}^e$$

The stiffness matrix of an elastic element

which, after comparison with the relation

$$\mathbf{f}^e = \mathbf{K}^e \mathbf{u}^e$$

gives us the equation determining coefficients of the element stiffness matrix

$$\mathbf{K}^e = \int_{\mathcal{V}} \left(\mathbf{B}^e\right)^{\mathsf{T}} \mathbf{D}\mathbf{B}^e d\mathcal{V}$$

The stiffness matrix of an elastic element

Building the element stiffness matrix can be considerably easy if we note that this matrix divides into blocks

$$\mathbf{K}^e = \begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{ij} & \mathbf{K}_{ik} \\ \mathbf{K}_{ji} & \mathbf{K}_{jj} & \mathbf{K}_{jk} \\ \mathbf{K}_{ki} & \mathbf{K}_{kj} & \mathbf{K}_{kk} \end{bmatrix}$$

in which any of them, for example $\mathbf{K}_{ij}$, can be calculated from the equation

$$\mathbf{K}_{ij} = \int_{\mathcal{V}} \left(\mathbf{B}_i\right)^{\mathsf{T}} \mathbf{D}\mathbf{B}_j d\mathcal{V}$$

The stiffness matrix of an elastic element

$$\mathbf{K}_{ij} = (\mathbf{B}_i)^{\mathsf{T}} \mathbf{D}\mathbf{B}_j \int_{\mathcal{V}} d\mathcal{V} = (\mathbf{B}_i)^{\mathsf{T}} \mathbf{D}\mathbf{B}_j Ab =$$

$$= \frac{EAb}{1-\nu^2} \begin{bmatrix} b_i b_j + c_i c_j \dfrac{1-\nu}{2} & b_i c_j \nu + b_j c_i \dfrac{1-\nu}{2} \\ b_j c_i \nu + b_i c_j \dfrac{1-\nu}{2} & c_i c_j + b_i b_j \dfrac{1-\nu}{2} \end{bmatrix}$$

The above matrix is the stiffness matrix for plane stress. where A is the surface of a 2D element; b is the thickness of 2D element

The stiffness matrix of an elastic element

We obtain the block of the stiffness matrix for plane strain accepting the matrix of material constants according to equation $\mathbf{M}\,\boldsymbol{\alpha}_i = \boldsymbol{\delta}_i$

$$\mathbf{K}_{ij} = \frac{EAb}{(1+\nu)(1-2\nu)}\begin{bmatrix} (1-\nu)b_i b_j + c_i c_j \dfrac{1-2\nu}{2} & b_i c_j \nu + b_j c_i \dfrac{1-2\nu}{2} \\ b_j c_i \nu + b_i c_j \dfrac{1-2\nu}{2} & (1-\nu)c_i c_j + b_i b_j \dfrac{1-2\nu}{2} \end{bmatrix}$$

The stiffness matrix of an elastic element

Since the local coordinate system is assumed in such a way that its axes are parallel to the global coordinate system, then we do not have to transform the stiffness matrix

Element strain and stress

We also calculate element strains

$$\varepsilon_x = \sum_{n=i,j,k} b_n u_{nx} \qquad \varepsilon_y = \sum_{n=i,j,k} b_n u_{ny} \qquad \gamma_{xy} = \sum_{n=i,j,k} \left(c_n u_{nx} + b_n u_{ny} \right)$$

We see that components of the strain vector are constant within the element which is the consequence of the assumption of linear shape functions. This element is called CST (constant strain triangle)

Element strain and stress

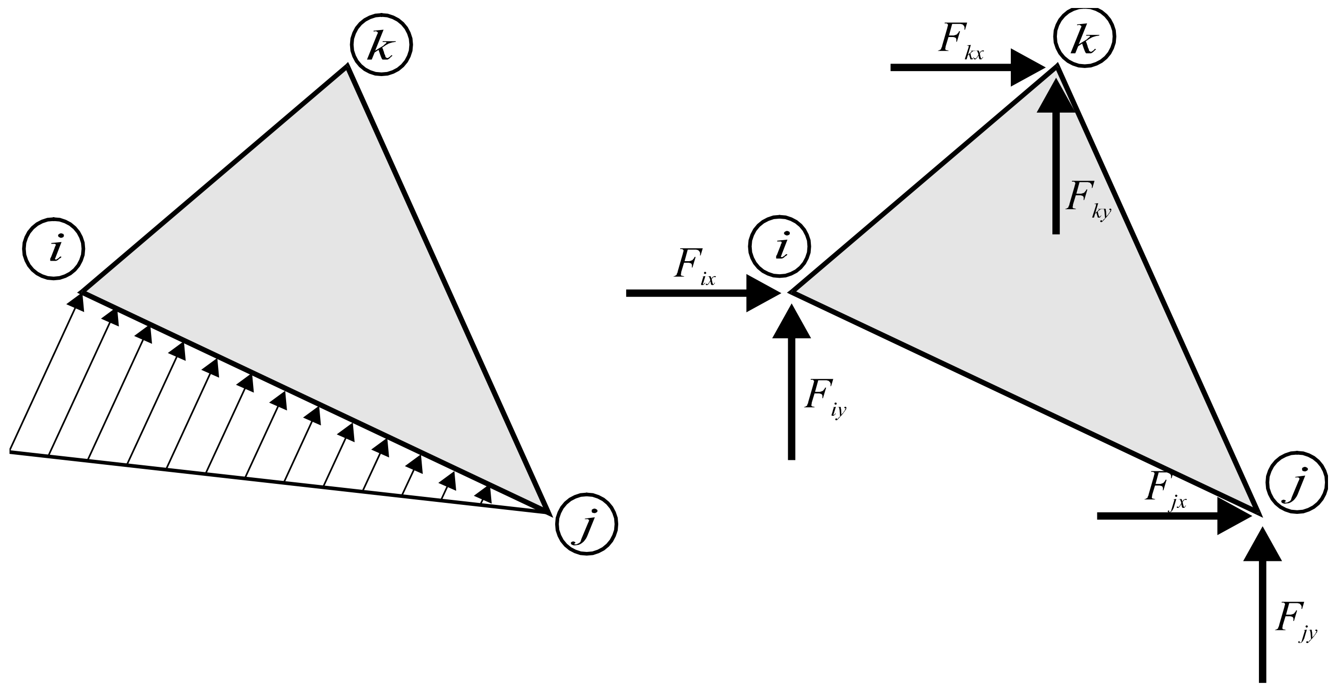
We determine element stresses from the constitutive equation $\sigma = D \cdot \varepsilon$ and equation $\mathbf{N}^e(x,y) = \left[N_i(x,y)\,\mathbf{I} \quad N_j(x,y)\,\mathbf{I} \quad N_k(x,y)\,\mathbf{I} \right]$ or $\boldsymbol{\sigma} = \mathbf{D} \cdot \boldsymbol{\varepsilon}$ according to the kind of variant that we deal with. It is obvious that strains, just as stresses are constant within the CST element

A Nodal force vector for a distributed load

Loads on 2D elements can be treated as loads on plane trusses which means that they can be applied to the nodes of a structure. But if a distributed load acting on the boundary of an element is given, then it should be converted to concentrated forces acting on the nodes of an element

A Nodal force vector for a distributed load

Nodal forces representing continuos loads



A Nodal force vector for a distributed load

Similarly, as in previously we apply the principal of virtual work giving the following equilibrium equation for this case

$$\left(\mathbf{u}^{e}\right)^{\mathsf{T}}\mathbf{f}^{e}+L_{ij}\int_{0}^{1}\mathbf{u}\left(\xi\right)^{\mathsf{T}}\mathbf{q}\left(\xi\right)d\xi=0$$

A Nodal force vector for a distributed load

$\mathbf{u}(\xi)$ - contains functions describing the displacement of the loaded edge

$\mathbf{q}(\xi)=\begin{bmatrix} q_x(\xi) \\ q_y(\xi) \end{bmatrix}$ - contains functions describing the load on the edge

L_{ij} - length of the edge

ξ - non-dimensional coordinate taking zero value at the node *i* and value 1 at the node *j*

then we write the vector $\mathbf{u}(\xi)$ as follows: $\mathbf{u}(\xi)=\mathbf{N}_{ij}^{e}\mathbf{u}^{e}$

A Nodal force vector for a distributed load

$\mathbf{N}^{e}_{ij}$ - matrix of shape functions for displacements of the boundary

$$\mathbf{N}^{e}_{ij} = \left[N^{o}_{i}\left(\xi\right) \mathbf{I} \quad N^{o}_{j}\left(\xi\right) \mathbf{I} \quad N^{o}_{k}\left(\xi\right) \mathbf{0} \right]$$

where $N^{o}_{i}(\xi) = 1 - \xi, \; N^{o}_{j}(\xi) = \xi$

A Nodal force vector for a distributed load

or in the developed form

$$\mathbf{N}_{ij}^{e} = \begin{bmatrix} 1-\xi & 0 & \xi & 0 & 0 & 0 \\ 0 & 1-\xi & 0 & \xi & 0 & 0 \end{bmatrix}$$

A Nodal force vector for a distributed load

After taking into consideration the shape functions, we obtain

$$\mathbf{f}^e = -L_{ij}\int_0^1 \begin{bmatrix} (1-\xi)q_x(\xi) \\ (1-\xi)q_y(\xi) \\ \xi q_x(\xi) \\ \xi q_y(\xi) \\ 0 \\ 0 \end{bmatrix} d\xi$$

A Nodal force vector for a distributed load

For example, let us calculate the nodal force vector due to the linear distributed load on the edge *i-j* of value q_{ix}, q_{iy} - at the node i and q_{jx}, q_{jy} - at the node j. We write such a load with the help of a non-dimensional coordinate ξ

$$\mathbf{q}(\xi) = \begin{bmatrix} q_{ix}(1-\xi) + q_{jx}\xi \\ q_{iy}(1-\xi) + q_{jy}\xi \end{bmatrix}$$

A Nodal force vector for a distributed load

and after inserting the above equation, we obtain

$$\mathbf{f}^e = -L_{ij}\begin{bmatrix} q_{ix}\int_0^1 (1-\xi)^2 d\xi + q_{jx}\int_0^1 (1-\xi)\xi d\xi \\ q_{iy}\int_0^1 (1-\xi)^2 d\xi + q_{jy}\int_0^1 (1-\xi)\xi d\xi \\ q_{ix}\int_0^1 (1-\xi)\xi d\xi + q_{jx}\int_0^1 \xi^2 d\xi \\ q_{iy}\int_0^1 (1-\xi)\xi d\xi + q_{jy}\int_0^1 \xi^2 d\xi \\ 0 \\ 0 \end{bmatrix}$$

A Nodal force vector for a distributed load

which after integration gives

$$\mathbf{f}^{e} = -\frac{L_{ij}}{6}\begin{bmatrix} 2q_{ix} + q_{jx} \\ 2q_{iy} + q_{jy} \\ q_{ix} + 2q_{jx} \\ q_{iy} + 2q_{jy} \\ 0 \\ 0 \end{bmatrix}$$

A Nodal force vector for a distributed load

For a particular case when the load is constant and equal to $\mathbf{q}(\xi) = \begin{bmatrix} q_{ox} \\ q_{oy} \end{bmatrix}$ we obtain

$$\mathbf{f}^{e} = -\frac{L_{ij}}{2} \begin{bmatrix} q_{ox} \\ q_{oy} \\ q_{ox} \\ q_{oy} \\ 0 \\ 0 \end{bmatrix}$$

A Nodal force vector for a distributed load

It should be remembered that the calculated forces are forces acting on the element. We obtain the necessary nodal forces changing the sense of vectors which means

$$\mathbf{p}^{e} = -\mathbf{f}^{e}$$

where $\mathbf{p}^{e}$ is the nodal force vector for the nodes touching the element e

A Nodal force vector due to a temperature load

As in the previous section, we apply the principal of virtual work to calculate alternative nodal forces replacing a temperature load. In accordance with the features of a CST element we will take into consideration only a constant temperature field within the element

A Nodal force vector due to a temperature load

The suitable equation of virtual work has the form

$$\left(\mathbf{u}^e\right)^{\mathsf{T}} \mathbf{f}^{et} = \int_{\mathcal{V}} \varepsilon^{\mathsf{T}} \sigma_t d\mathcal{V} = \int_{\mathcal{V}} \varepsilon^{\mathsf{T}} \mathbf{D} \varepsilon_t d\mathcal{V}$$

$\boldsymbol{\sigma}_t$ - stress field in the element which is caused by the temperature

$\boldsymbol{\varepsilon}_t$ - strain of the element caused by the change of a temperature

A Nodal force vector due to a temperature load

Assuming isotropy of a 2D element we obtain

$$\varepsilon_t = \alpha_t \Delta t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

After inserting geometric relation

$$\mathbf{f}^{et} = \alpha_t \Delta t \int_{\mathcal{V}} \left(\mathbf{B}^e\right)^{\mathsf{T}} \mathbf{D} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} d\mathcal{V} = \alpha_t \Delta t A b \left(\mathbf{B}^e\right)^{\mathsf{T}} \mathbf{D} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

A Nodal force vector due to a temperature load

For a plane stress problem this equation is simplified to the following relation

$$\mathbf{f}_{\mathrm{PSN}}^{et} == \frac{\alpha_t \Delta t E A b}{1-\nu} \begin{bmatrix} b_i \\ c_i \\ b_j \\ c_j \\ b_k \\ c_k \end{bmatrix}$$

where b_i ... c_k are coefficients of shape functions of the CST element

A Nodal force vector due to a temperature load

Plane strain gives a slightly different nodal force vector

$$\mathbf{f}_{\mathrm{PSO}}^{et} == \frac{\alpha_t \Delta t E A b}{(1+\nu)(1-2\nu)} \begin{bmatrix} b_i \\ c_i \\ b_j \\ c_j \\ b_k \\ c_k \end{bmatrix}$$

A Nodal force vector due to a temperature load

As in previous sections, we should change the signs of components of nodal forces before applying them to the nodes

$$\mathbf{p}^{et} = -\mathbf{f}^{et}$$

A Nodal force vector due to a temperature load

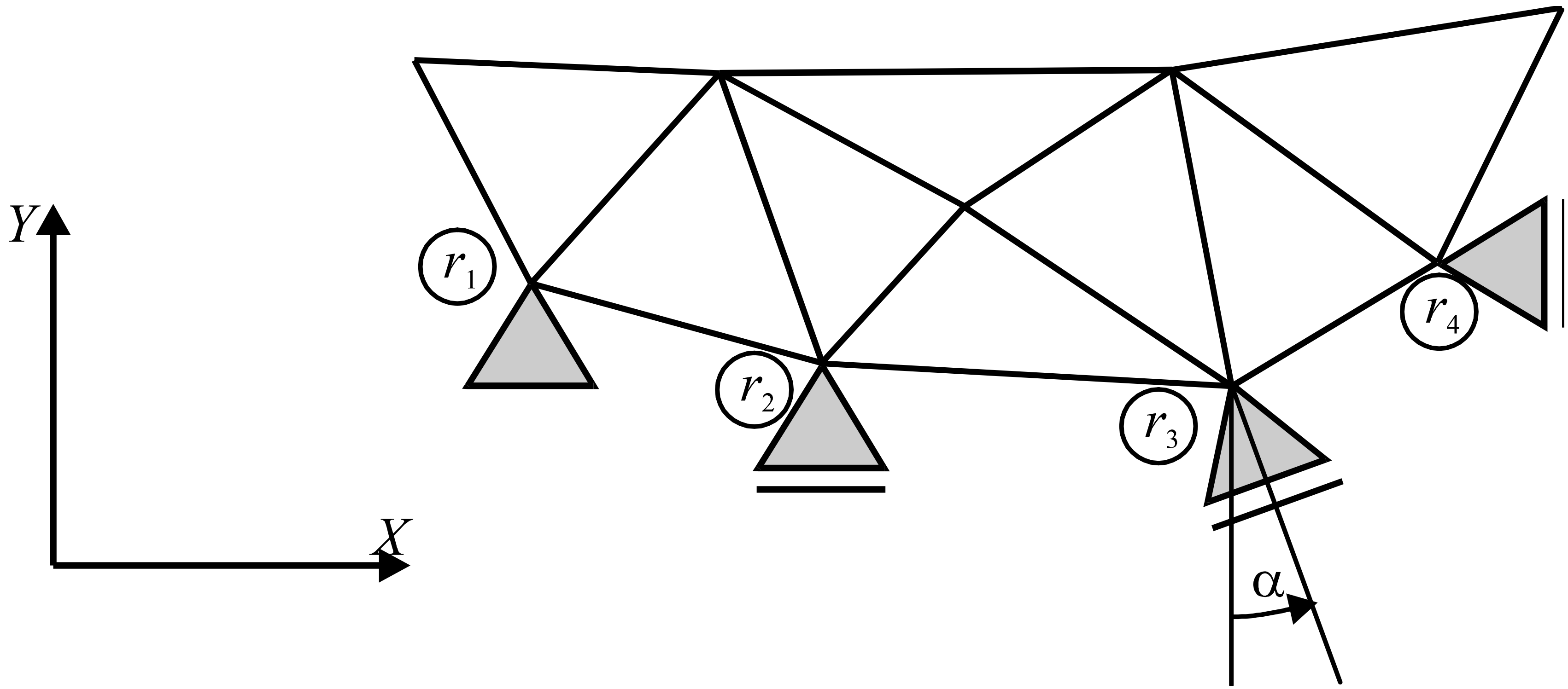
We calculate stresses in the element undergoing the action of a temperature taking into consideration strains caused by the thermal expansion of the element

$$\boldsymbol{\sigma}_t = \mathbf{D}\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_t\right) = \mathbf{D}\left(\mathbf{B}\,\mathbf{u}^e - \alpha_t \Delta t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$$

Boundary conditions of a 2D element

Boundary conditions of a two-dimensional structure can be treated analogously to the conditions in a plane truss because the nodes of both systems have two degrees of freedom on the XY plane

Boundary conditions of a 2D element



Boundary conditions of a 2D element

Hence we have: fixed supports (at the node r_1) and supports which can move along the X axis (at the node r_2), next supports which can move along the Y axis (at the node r_4) or skew supports (at the node r_3).

Boundary conditions of a 2D element

The boundary conditions for these supports are as follows

node r_1: $u_{r_1 X} = 0, \; u_{r_1 Y} = 0,$

node r_2: $u_{r_2 Y} = 0 \; ,$

node r_4: $u_{r_4 X} = 0,$

for node r_3, where constraints are not consistent with the axes of the global coordinate system we propose the use of boundary elements described in Chapter 2.