Finite Element Method
Statics of plates

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Introduction

Plates are one of the most commonly used elements in structures. They can be found in almost every building or mechanical structure. The geometric shape of a plate can be defined similarly to a 2D element, but they differ in the way of loading. Plates are loaded with normal loads to their surfaces which cause bending. Bending is not present in the case of the deformation of the 2D element

Introduction

Analytical methods of determining both deflections and internal forces were described by Euler, Bernoulli, Germain, Lagrange, Poisson and especially by Navier in papers which appeared at the end of the 18th century described by Rao (1982)

Introduction

Many important statics and dynamics problems of plates were solved by analytical methods (mainly by the method of the Fourier series), but they are inaccurate both in the case of problems with complex boundary conditions and complicated shapes of plates. However, the finite element method has proved to be universal and although it gives approximate solutions, they are precise enough for practical applications

equations of the classic theory of plates

We assume that these plates the assumptions of the classic theory of thin plates

a) thickness of a plate is small in comparison with its other dimensions;

b) deflections of plates are small in comparison with its thickness;

c) middle plane does not undergo lengthening (or shortening);

equations of the classic theory of plates

d) points lying on the lines which are perpendicular to the middle plane before its deformation lie on these lines after the deformation;

e) components of stress which are perpendicular to the plane of the plate can be neglected

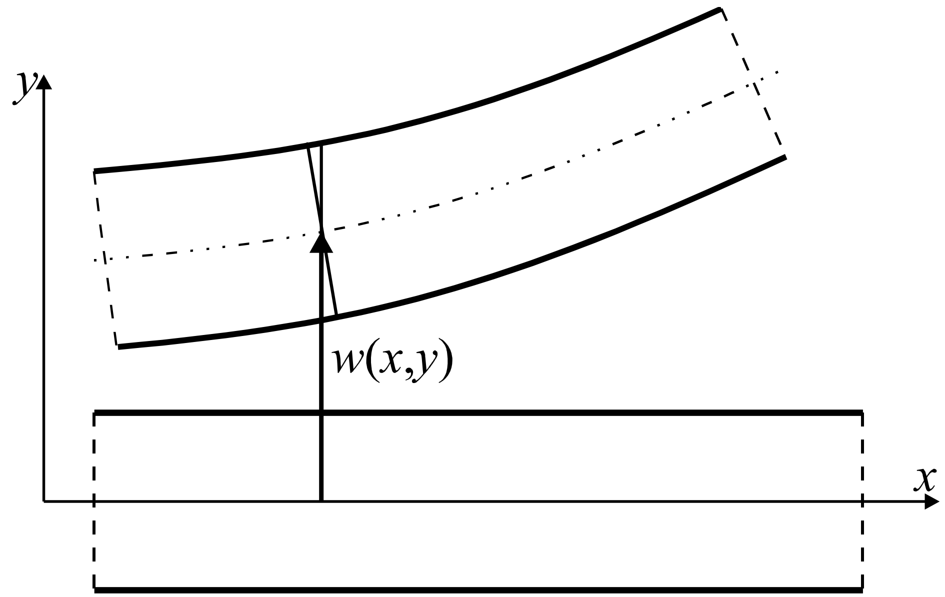
equations of the classic theory of plates

From point d) of the above assumptions it follows that the displacement of points lying within the plate varies linearly with its thickness

$$u_x = -z\frac{\partial w}{\partial x}$$

$$u_y = -z\frac{\partial w}{\partial y}$$

$$u_z = w(x, y)$$



equations of the classic theory of plates

Thus stains are expressed by the relations

$$\varepsilon_x = \frac{\partial u_x}{\partial x} = -z\frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_y = \frac{\partial u_y}{\partial y} = -z\frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y}$$

equations of the classic theory of plates

The strain vector can be presented in the form

$$\boldsymbol{\varepsilon} = -z\,\partial w(x, y)$$

where vector ∂ is the vector of differential operators

$$\partial = \begin{bmatrix} \partial_{xx} \\ \partial_{yy} \\ 2\partial_{xy} \end{bmatrix} \qquad \partial_{xx} = \frac{\partial^2}{\partial x^2} \qquad \partial_{yy} = \frac{\partial^2}{\partial y^2} \qquad \partial_{xy} = \frac{\partial^2}{\partial x \partial y}$$

equations of the classic theory of plates

Let us assume that there is a plane stress condition in the plate, so the stress vector can be determined as follows

$$\boldsymbol{\sigma} = \mathbf{D} \cdot \boldsymbol{\varepsilon} = -z\,\mathbf{D}\,\partial w(x, y)$$

where $\mathbf{D}$ is the matrix of material constants determined for plane stress

equations of the classic theory of plates

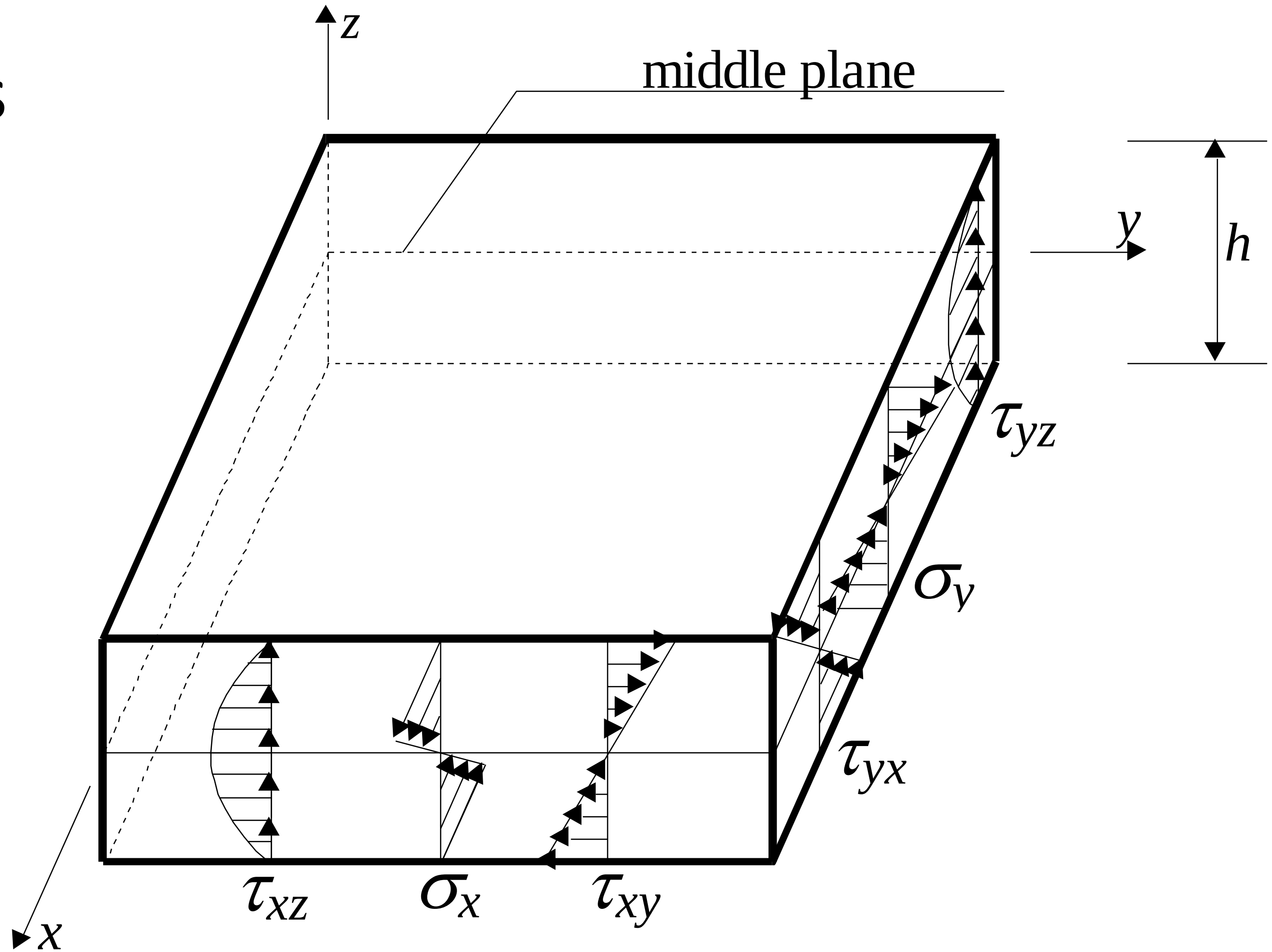
Now we introduce in the expression of internal forces (moments and shearing forces)

$$M_x = \int_{-h/2}^{h/2} \sigma_x z dz \qquad M_y = \int_{-h/2}^{h/2} \sigma_y z dz \qquad M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z dz$$

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz} dz \qquad Q_y = \int_{-h/2}^{h/2} \tau_{yz} dz$$

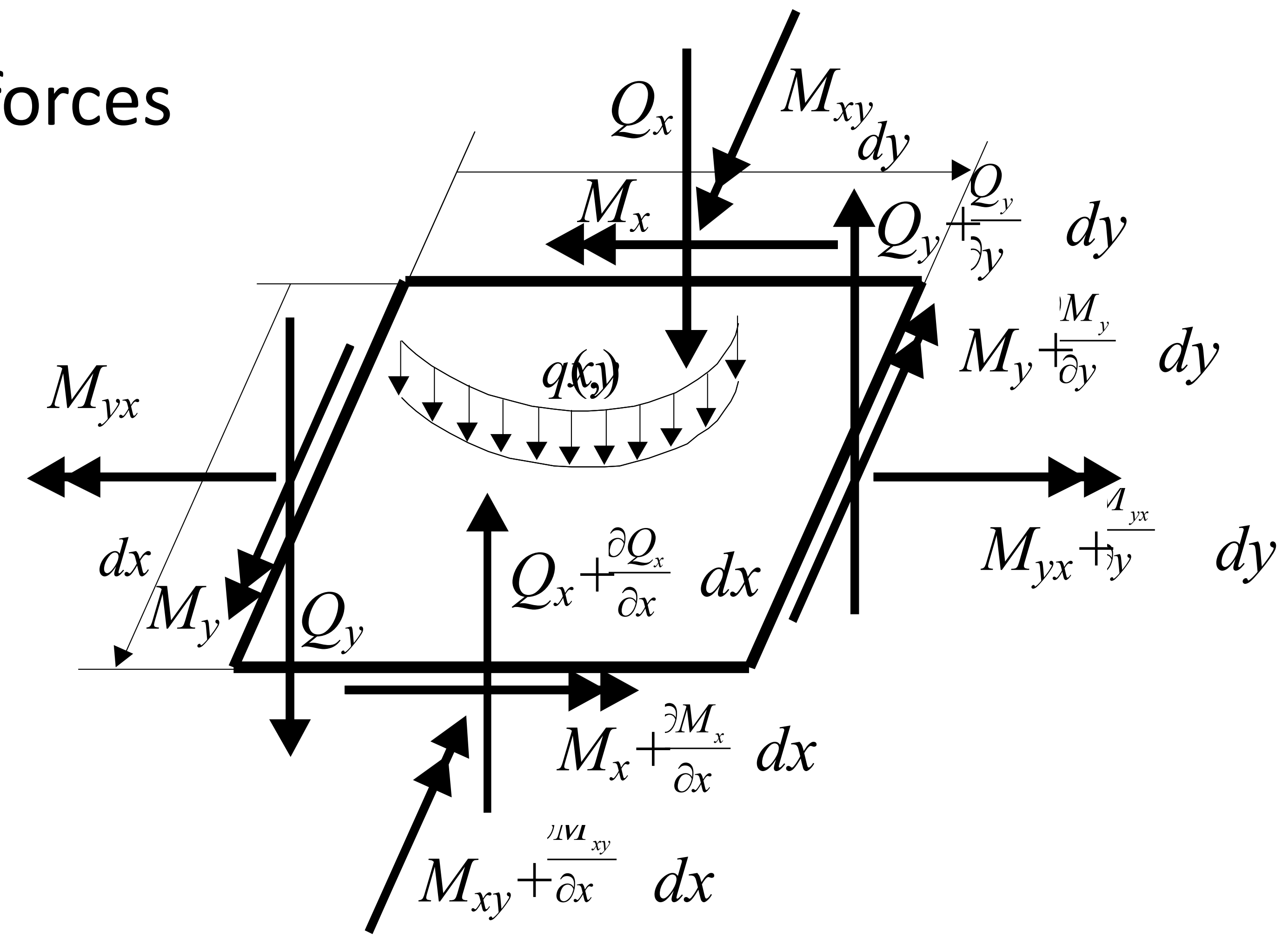
equations of the classic theory of plates

stresses



equations of the classic theory of plates

internal forces



equations of the classic theory of plates

The equilibrium of an infinitesimal plate element leads to the set of equations

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q(x,y) = 0$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = Q_x$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = Q_y$$

equations of the classic theory of plates

After integration we obtain

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$M_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

where D denotes the plate stiffness defined by the equation

$$D = \frac{Eh^3}{12\left(1-\nu^2\right)}$$

equations of the classic theory of plates

we also obtain relations for the shearing forces

$$Q_x = -D\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right)$$

$$Q_y = -D\left(\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3}\right)$$

equations of the classic theory of plates

after Inserting equation describing shearing forces we obtain

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x,y)}{D}$$

equations of the classic theory of plates

It is a biharmonic partial differential equation which should be satisfied by the function of deflection w(x,y) within the plate. The following boundary conditions should be realised at the edges of the plate

equations of the classic theory of plates

$$a)\ \ w=0,\ \ \frac{\partial w}{\partial n}=0\ -\text{on the fixed edge}$$

$$b)\ \ w=0,\ \ \frac{\partial^2 w}{\partial n^2}=0\ -\text{on the free supported edge}$$

$$c)\ \ M_n=0,\ \ V_n=0\ -\text{on the free edge}$$

equations of the classic theory of plates

In the above equations n defines the direction of the line which is perpendicular to the edge and V_n is the reduced force. Siła ta łączy wpływ momentu skręcającego M_{ns} oraz siły poprzecznej Q_n na brzegu swobodnym

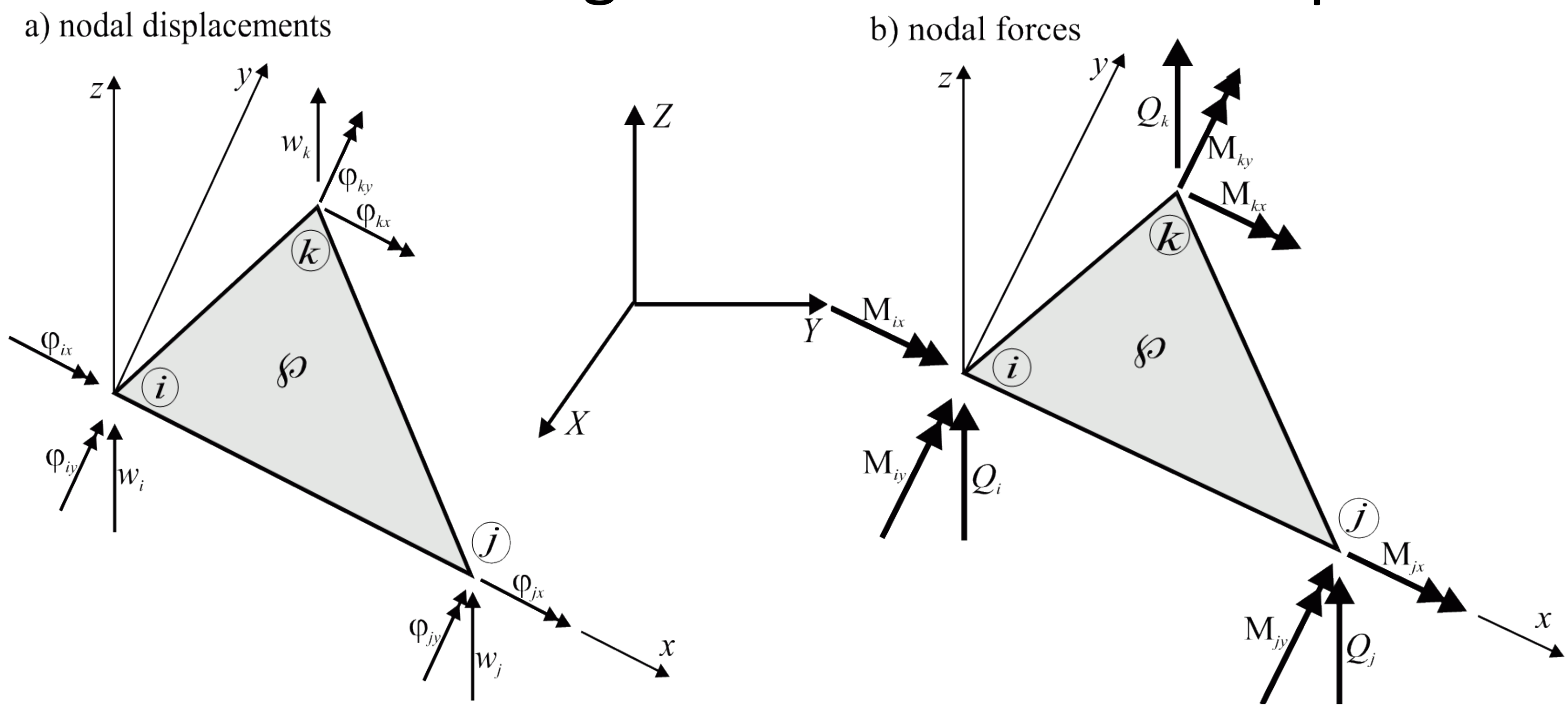
$$V_n = Q_n - \frac{\partial M_{ns}}{\partial s} = -D\left[\frac{\partial^3 w}{\partial n^3} + (2-\nu)\frac{\partial^3 w}{\partial n \partial s^2}\right]$$

The modification of the boundary conditions is necessary here because the fourth order cannot be solved for three boundary conditions coming from the requirement of zero stress on the free edge:

$$M_{ns} = 0, M_n = 0, Q_n = 0.$$

A finite triangular element of a thin plate

Now we show the way of building the stiffness matrix of a triangular element of a thin plate



A finite triangular element of a thin plate

We also introduce a few convenient notations

$w(x, y)$ - stands for the function of displacement of the middle plane of an element;

$\varphi_x = \dfrac{\partial w}{\partial y}$ - rotation angle of the element about the x axis;

$\varphi_y = -\dfrac{\partial w}{\partial x}$ - rotation angle of the element about the y axis.

A finite triangular element of a thin plate

As seen, the node of a plate element has three degrees of freedom. Hence nodal displacement vectors of the element in the local system can be written as follows

$$\mathbf{u}'_i = \begin{bmatrix} w_i \\ \varphi_{ix} \\ \varphi_{iy} \end{bmatrix} \qquad \mathbf{u}'_j = \begin{bmatrix} w_j \\ \varphi_{jx} \\ \varphi_{jy} \end{bmatrix} \qquad \mathbf{u}'_k = \begin{bmatrix} w_k \\ \varphi_{kx} \\ \varphi_{ky} \end{bmatrix}$$

and an element displacement vector $\mathbf{u}'^e = \begin{bmatrix} \mathbf{u}'_i \\ \mathbf{u}'_j \\ \mathbf{u}'_k \end{bmatrix}$

A finite triangular element of a thin plate

Directions of both nodal displacements and forces are the same, so the nodal forces vectors have a similar notation

$$\mathbf{f}_i' = \begin{bmatrix} Q_i \\ M_{ix} \\ M_{iy} \end{bmatrix} \qquad \mathbf{f}_j' = \begin{bmatrix} Q_j \\ M_{jx} \\ M_{jy} \end{bmatrix} \qquad \mathbf{f}_k' = \begin{bmatrix} Q_k \\ M_{kx} \\ M_{ky} \end{bmatrix}$$

Hence we write the nodal force vector of the element as follows

$$\mathbf{f}'^e = \begin{bmatrix} \mathbf{f}_i' \\ \mathbf{f}_j' \\ \mathbf{f}_k' \end{bmatrix}$$

A finite triangular element of a thin plate

We approximate the surface of the deformed element by the polynomial of the third order proposed by J.LTocher in 1962

$$w(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^3 + a_8\left(x^2 y + xy^2\right) + a_9 y^3 = \boldsymbol{\eta}^{\mathrm{T}}\mathbf{a}$$

where

$$\eta = \begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \\ x^3 \\ x^2 y + xy^2 \\ y^3 \end{bmatrix} \qquad \mathrm{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix}$$

A finite triangular element of a thin plate

We determine the coefficients a_1 ... a_9 of the function $w(x,y)$ from the boundary conditions at the nodes i, j, k

$$w(x_i, y_i) = w_i \qquad \varphi_x(x_i, y_i) = \varphi_{ix} \qquad \varphi_y(x_i, y_i) = \varphi_{iy}$$

$$w(x_j, y_j) = w_j \qquad \varphi_x(x_j, y_j) = \varphi_{jx} \qquad \varphi_y(x_j, y_j) = \varphi_{jy}$$

$$w(x_k, y_k) = w_k \qquad \varphi_x(x_k, y_k) = \varphi_{kx} \qquad \varphi_y(x_k, y_k) = \varphi_{ky}$$

A finite triangular element of a thin plate

After calculating the rotation angles, we obtain

$$\varphi_x = \frac{\partial w(x,y)}{\partial y} = a_3 + a_5 x + 2a_6 y + a_8(x^2 + 2xy) + 3a_9 y^2$$

$$\varphi_y = -\frac{\partial w(x,y)}{\partial x} = -\left[a_2 + 2a_4 x + a_5 y + 3a_7 x^2 + a_8(2xy + y^2)\right]$$

A finite triangular element of a thin plate

Now we insert equation into boundary conditions obtaining

$$\mathbf{M}\mathbf{a} = \mathbf{u}'^{e}$$

where $\mathbf{M}$ is the square matrix dependent on nodal coordinates of the element

A finite triangular element of a thin plate

$$
\mathbf{M}=
\begin{array}{c}
\begin{array}{ccccccccc}
A_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9
\end{array}
\\
\left[\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & x_j & 0 & x_j^2 & 0 & 0 & x_j^3 & 0 & 0 \\
0 & 0 & 1 & 0 & x_j & 0 & 0 & x_j^2 & 0 \\
0 & -1 & 0 & -2x_j & 0 & 0 & -3x_j^2 & 0 & 0 \\
1 & x_k & y_k & x_k^2 & x_k y_k & y_k^2 & x_k^3 & x_k^2 y_k + x_k y_k^2 & y_k^3 \\
0 & 0 & 1 & 0 & x_k & 2y_k & 0 & x_k^2 + 2x_k y_k & 3y_k^2 \\
0 & -1 & 0 & -2x_k & -y_k & 0 & -3x_k^2 & -2x_k y_k - y_k^2 & 0
\end{array}\right]
\end{array}
\begin{array}{c}
\\
w_i \\ \varphi_{ix} \\ \varphi_{iy} \\ w_j \\ \varphi_{jx} \\ \varphi_{jy} \\ w_k \\ \varphi_{kx} \\ \varphi_{ky}
\end{array}
$$

A finite triangular element of a thin plate

We can present the solution of equation $\mathbf{M}\,\mathbf{a} = \mathbf{u}'^{e}$ as follows $\mathbf{a} = \mathbf{M}^{-1}\,\mathbf{u}'^{e}$

where $\mathbf{M}^{-1}$ is the inverse matrix of $\mathbf{M}$. The solution of $\mathbf{M}^{-1}$ is possible when which is not always the case in our problem because $\det \mathbf{M} = x_j^5 y_k^5 \left(2x_k + y_k - x_j\right)$

A finite triangular element of a thin plate

It means that in cases when the node k of the element is on the line described by equation , then the matrix $\mathbf{M}$ is singular. Thus, the problem is solved by changing the local coordinate system

Now we calculate a strain vector

$$\boldsymbol{\varepsilon} = -z\partial w(x, y) = -z\partial\boldsymbol{\eta}^{\mathsf{T}}\mathbf{M}^{-1}\mathbf{u}'^{e} = -z\mathbf{B}^{*}\mathbf{M}^{-1}\mathbf{u}'^{e}$$

A finite triangular element of a thin plate

Hence we can make use of the definition of the stiffness matrix

$$\mathbf{K}'^{\,e} = \int_{\mho} \left(\mathbf{B}^{e}\right)^{\mathsf{T}} \mathbf{D}\mathbf{B}^{e} d\mho = \left(\mathbf{M}^{-1}\right)^{\mathsf{T}} \int_{-h/2}^{h/2} z^{2} dz \int_{\mathrm{A}} \left(\mathbf{B}^{*}\right)^{\mathsf{T}} \mathbf{D}\mathbf{B}^{*} d\mathrm{A}\mathbf{M}^{-1} =$$

$$= \frac{Eh^{3}}{12\left(1-\nu^{2}\right)} \left(\mathbf{M}^{-1}\right)^{\mathsf{T}} \int_{\mathrm{A}} \left(\mathbf{B}^{*}\right)^{\mathsf{T}} \mathbf{D}\mathbf{B}^{*} d\mathrm{A}\mathbf{M}^{-1}$$

A finite triangular element of a thin plate

After denoting the integration in the above equation by and applying the definition of plate stiffness we have

$$\mathbf{K}'^{\,e} = D\left(\mathbf{M}^{-1}\right)^{\mathsf{T}} \mathbf{K}^{*} \mathbf{M}^{-1}$$

A finite triangular element of a thin plate

After calculating the matrix multiplication inside the integration, we have

$$\mathbf{K}^* = \int_A \mathbf{S} d\mathrm{A}$$

A finite triangular element of a thin plate

where

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 4\nu & 12x & 4(y+x\nu) & 12y\nu \\ 0 & 0 & 0 & 0 & 2(1-\nu) & 0 & 0 & 4(x+y)(1-\nu) & 0 \\ 0 & 0 & 0 & 4\nu & 0 & 4 & 12x\nu & 4(y\nu+x) & 12y \\ 0 & 0 & 0 & 12x & 0 & 12x\nu & 36x^2 & 12(xy+x^2\nu) & 36xy\nu \\ 0 & 0 & 0 & 4(y+x\nu) & 4(x+y)(1-\nu) & 4(y\nu+x) & 12(xy+x^2\nu) & \begin{array}{c} 4(3y^2+4xy+3x^2)+ \\ -8\nu(x^2+xy+y^2) \end{array} & 12(y^2\nu+xy) \\ 0 & 0 & 0 & 12y\nu & 0 & 12y & 36xy\nu & 12(y^2\nu+xy) & 36y^2 \end{bmatrix}$$

A finite triangular element of a thin plate

- While calculating the integration of functions, the following relations are helpful

$$\int_A dA = \frac{1}{2} x_j y_k$$

$$\int_A x dA = \frac{1}{6} x_j y_k \left(x_j + x_k \right)$$

$$\int_A y dA = \frac{1}{6} x_j y_k^2$$

$$\int_A x^2 dA = \frac{1}{12} x_j y_k \left(x_j^2 + x_j x_k + x_k^2 \right)$$

$$\int_A xy dA = \frac{1}{24} x_j y_k^2 \left(x_j + 2x_k \right)$$

$$\int_A y^2 dA = \frac{1}{12} x_j y_k^2$$

A finite triangular element of a thin plate

The rotation matrix of an element $\mathbf{R}^e$ is equal to

$$\mathbf{R}^e = \begin{bmatrix} \mathbf{R}_i & & \\ & \mathbf{R}_j & \\ & & \mathbf{R}_k \end{bmatrix}$$

where $\mathbf{R}_i$, $\mathbf{R}_j$, $\mathbf{R}_k$ are the transformation matrices of nodes. If we use the same coordinate systems for all nodes, then we can use only one transformation matrix: $\mathbf{R}_j = \mathbf{R}_i$, $\mathbf{R}_k = \mathbf{R}_i$,

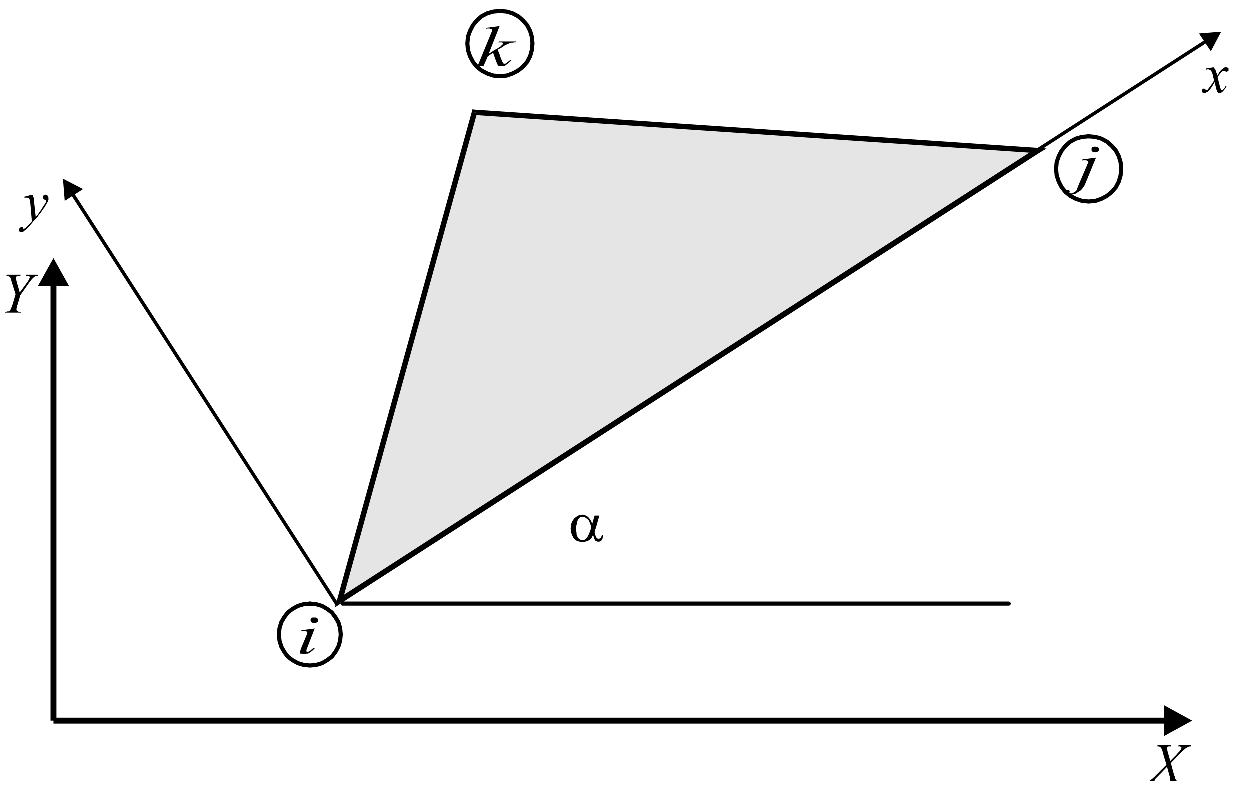
A finite triangular element of a thin plate

$$\mathbf{R}_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}$$

where , and α is the angle between the X axis of the global system and the x axis of the local system. Value 1 in the first row of the matrix $\mathbf{R}_\mathrm{i}$ is the consequence of a fact that axes Z and z are parallel

A finite triangular element of a thin plate

The plate arrangement in the global coordinate system



A finite triangular element of a thin plate

The triangular element for which the matrix stiffness has been obtained has a convenient feature. Namely, it allows us to discrete plates of any shape without any difficulty. This element joined with a 2D triangular element can be used as a shell element

A triangular element of a thin shell

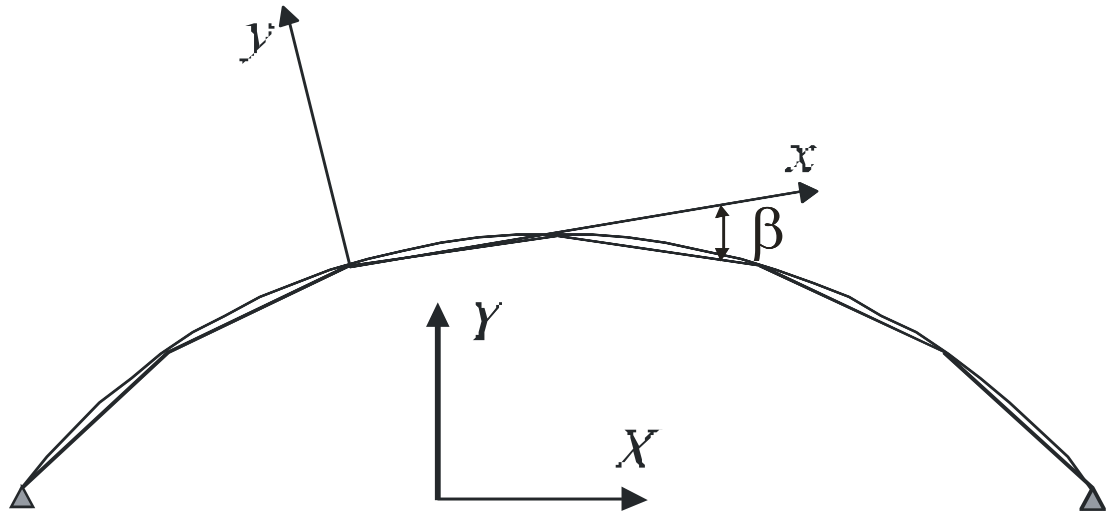
As it has been noted at the previous point, an element containing 2D triangular and plate elements can be used as a shell element. Approximating a curved surface (which is the middle surface of a shell) with the help of plate elements reminds the simplification we apply to approach the arc with the help of a broken line.

A triangular element of a thin shell

We intuitively feel that the smaller the curve line segments are, the better they replace the curve axis of the arc

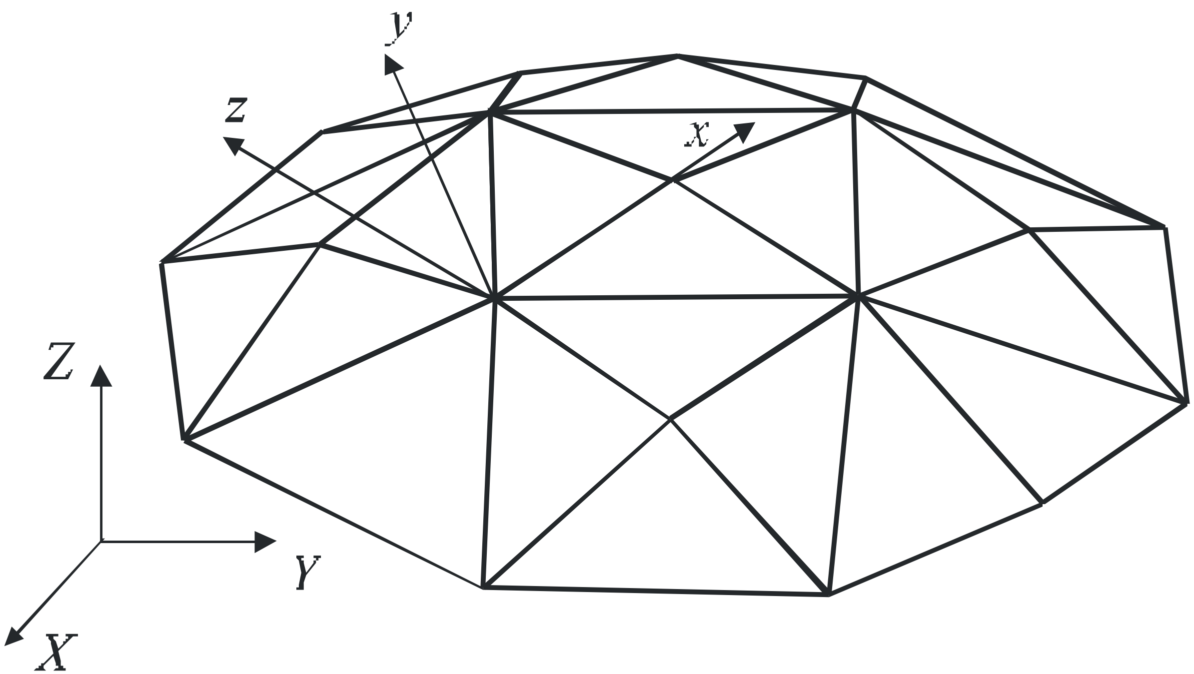
A triangular element of a thin shell

Similarly the smaller the plane shell element dimensions and the smaller β angles of neighbouring elements are, the better this element describes displacements and internal forces in the structure



A triangular element of a thin shell

The exemplary shell division into finite elements



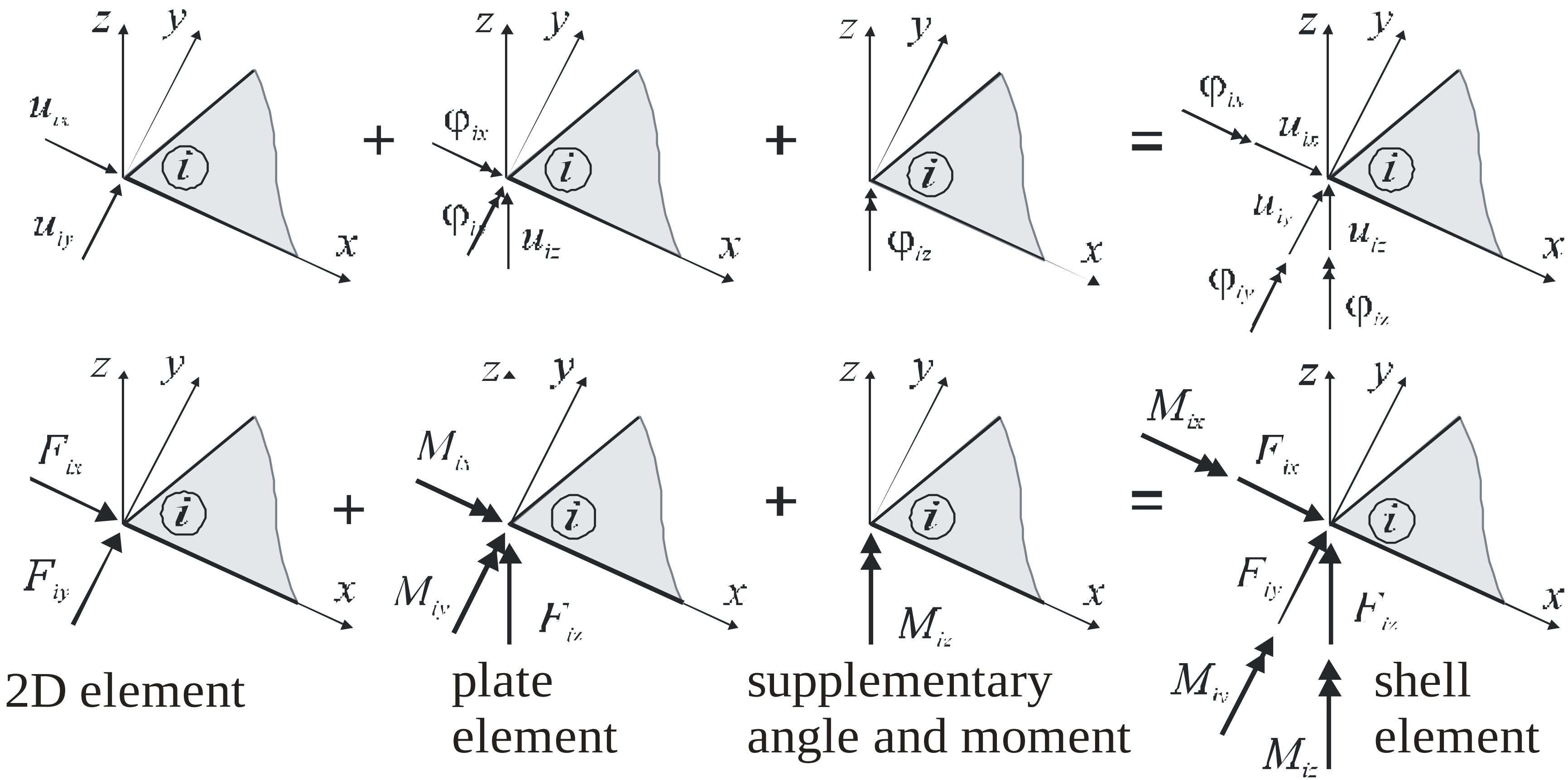
A triangular element of a thin shell

Connecting displacement and internal force vectors of the triangular elements described previously, we obtain shell element nodes possessing five degrees of freedom

$$\mathbf{u}_i' = \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{iz} \\ \phi_{ix} \\ \phi_{iy} \end{bmatrix} \qquad \mathbf{f}_i' = \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{iz} \\ M_{ix} \\ M_{iy} \end{bmatrix}$$

A triangular element of a thin shell

The shell element composition of a 2D and plate elements



A triangular element of a thin shell

Simplifying the description of a node movement by disregarding the rotation around the axis perpendicular to the element leads to the singularity of the shell stiffness matrix modelled by the elements mentioned before

A triangular element of a thin shell

This difficulty is solved by assuming three components of the rotation and moment vectors which requires the evaluation of the plate element torsional stiffness.

A triangular element of a thin shell

Since the torsional stiffness is not important in shell statics and dynamics problems, the fictitious value of this stiffness is often assumed

A triangular element of a thin shell

Hence the dependence between the torsional moments and angles can be presented as a variable independent of other nodal forces and displacements of an element

A triangular element of a thin shell

$$\begin{bmatrix} M_{iz} \\ M_{jz} \\ M_{kz} \end{bmatrix} = \alpha E h A \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 1 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{iz} \\ \varphi_{jz} \\ \varphi_{kz} \end{bmatrix}$$

In the above relationship suggested by Zienkiewicz (1972), E is Young's modulus, h is the element thickness, A is the area of a cross section and α denotes an indemensional coefficient which is so small that it does not have any significant influence on the solution of a set of equations

A triangular element of a thin shell

we obtain the stiffness matrix of the triangular shell element nodes having six degrees of freedom

$$\mathbf{f}'^e = \mathbf{K}'^e \mathbf{u}'^e$$

$$\begin{bmatrix} \mathbf{f}'_i \\ \mathbf{f}'_j \\ \mathbf{f}'_k \end{bmatrix} = \begin{bmatrix} \mathbf{K}'_{ii} & \mathbf{K}'_{ij} & \mathbf{K}'_{ik} \\ \mathbf{K}'_{ji} & \mathbf{K}'_{jj} & \mathbf{K}'_{jk} \\ \mathbf{K}'_{ki} & \mathbf{K}'_{kj} & \mathbf{K}'_{kk} \end{bmatrix} \begin{bmatrix} \mathbf{u}'_i \\ \mathbf{u}'_j \\ \mathbf{u}'_k \end{bmatrix}$$

A triangular element of a thin shell

where $\mathbf{f}'_i$ and $\mathbf{u}'_i$ denote full vectors of nodal forces and displacements

$$\mathbf{u}'_i = \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{iz} \\ \phi_{ix} \\ \phi_{iy} \\ \phi_{iz} \end{bmatrix} \qquad \mathbf{f}'_i = \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \end{bmatrix}$$

A triangular element of a thin shell

Every block of the stiffness matrix consists of ‘2D element’, ‘plate’ and ‘torsional’ parts

$$\begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \end{bmatrix} = \left[\begin{array}{cc|ccc|c} \multicolumn{2}{c|}{\multirow{2}{*}{${}^{t}\mathbf{K}_{ij}$}} & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{3}{c|}{} & \\ 0 & 0 & \multicolumn{3}{c|}{{}^{p}\mathbf{K}_{ij}} & \\ 0 & 0 & \multicolumn{3}{c|}{} & \\ \hline 0 & 0 & 0 & -0.5\,a_s & -0.5\,a_s & a_s \end{array}\right] \begin{bmatrix} u_{jx} \\ u_{jy} \\ u_{jz} \\ \varphi_{jx} \\ \varphi_{jy} \\ \varphi_{jz} \end{bmatrix}$$

A triangular element of a thin shell

Transformation of this matrix to the global coordinate system can be done in the way described in Chapter 5 in which we present the transformation of the stifness matrix of a 3D frame element with nodes having six degrees of freedom just as the nodes of a shell element

A triangular element of a thin shell

The method of obtaining the components of the rotate matrix is suitable for the triangular shell element whose i and j nodes determine the direction of the local x axis and the third k node can be a directional point

A triangular element of a thin shell

The shell element described above is the simplest element which enables us to solve any shell statics problem. There certainly are more complex elements, both plane and space elements with at least four nodes described in books devoted to this subject

A triangular element of a thin shell

We must remember about the possibility of significant simplification of a shell element description in case of axisymmetric structures. It is also possible to use cone or curvelinear elements with nodes having three degrees of freedom (Rakowski and Kacprzyk (1993), Zienkiewicz (1972, 1994))