CHAPTER VII.
STATICS OF PLATES

Plates are one of the most often used elements in structures. They can be found in almost every building or mechanic structure. The geometric shape of a plate can be defined similarly to a 2D element (Chapter VI), but they differ in the way of loading. Plates are loaded with normal loads to their surfaces which cause their bending. Bending is not present in case of the deformation of the 2D element.

Analytic methods of the determination of both deflections and internal forces were described by Euler, Bernoulli, Germain, Lagrange, Poissona and specially by Navier in papers which appeared on the turn of the 18th century [16]. Literature devoted to the theory of plates is unusually reach, the books [9], [11], [18] can be recommended to interested readers.

Many important statics and dynamics problems of plates were solved by analytic methods (mainly by the method of the Fourier series), but they disappoint both in the case of problems with complex boundary conditions and complicated shapes of plates. However, the finite element method has proved to be universal and although it gives approximate solutions, they are precise enough for practical applications.

7.1. BASIC ASSUMPTIONS AND EQUATIONS OF THE CLASIC THEORY OF PLATE

We assume that plates which we will occupy with satisfy the assumptions of the classic theory of thin plates [18]:

a) thickness of a plate is small in comparison with its other dimensions;
b) deflections of plates are small in comparison with its thickness;
c) middle plate does not undergo lengthening (or shortening);
d) points lying on the lines which are perpendicular to the middle plate before its deformation lie on these lines after the deformation;
e) components of stress which are perpendicular to the plane of the plate can be neglected.

It comes from point d) of the above assumptions that the displacement of points lying within the plate changes linearly with its thickness Fig.7.1:

$$u_x = -z\frac{\partial w}{\partial x}\,,\ u_y = -z\frac{\partial w}{\partial y}\,,\ u_z = w(x,y)\,. \tag{7.1}$$

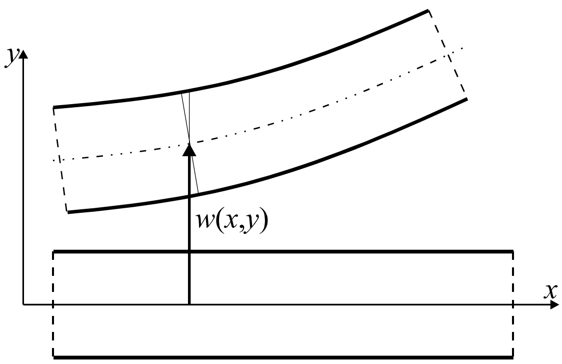

Fig.7.1

Thus stains are expressed by the relations:

$$\varepsilon_x = \frac{\partial u_x}{\partial x} = -z\frac{\partial^2 w}{\partial x^2},\ \varepsilon_y = \frac{\partial u_y}{\partial y} = -z\frac{\partial^2 w}{\partial y^2},\ \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y}. \tag{7.2}$$

The strain vector can be presented in the form:

$$\boldsymbol{\varepsilon} = -z\, \partial w(x,y), \tag{7.3}$$

where vector ∂ is the vector of differential operators:

$$\partial = \begin{bmatrix} \partial_{xx} \\ \partial_{yy} \\ 2\partial_{xy} \end{bmatrix},\ \text{a}\ \partial_{xx} = \frac{\partial^2}{\partial x^2},\ \partial_{yy} = \frac{\partial^2}{\partial y^2},\ \partial_{xy} = \frac{\partial^2}{\partial x \partial y}.$$

Let us assume that there is a plane stress in the plate, so the stress vector can be determined as follows:

$$\boldsymbol{\sigma} = \mathbf{D}\, \boldsymbol{\varepsilon} = -z\, \mathbf{D}\, \partial w(x,y), \tag{7.4}$$

where **D** is the matrix of material constants determined for plane stress (equation **Błąd! Nie można odnaleźć źródła odwołania.**)).

Now we lead in the expression of internal forces (moments and shearing forces - Fig.7.2)

$$M_x = \int_{-h/2}^{h/2} \sigma_x z dz,\ M_y = \int_{-h/2}^{h/2} \sigma_y z dz,\ M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z dz,$$

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz} dz,\ Q_y = \int_{-h/2}^{h/2} \tau_{yz} dz. \tag{7.5}$$

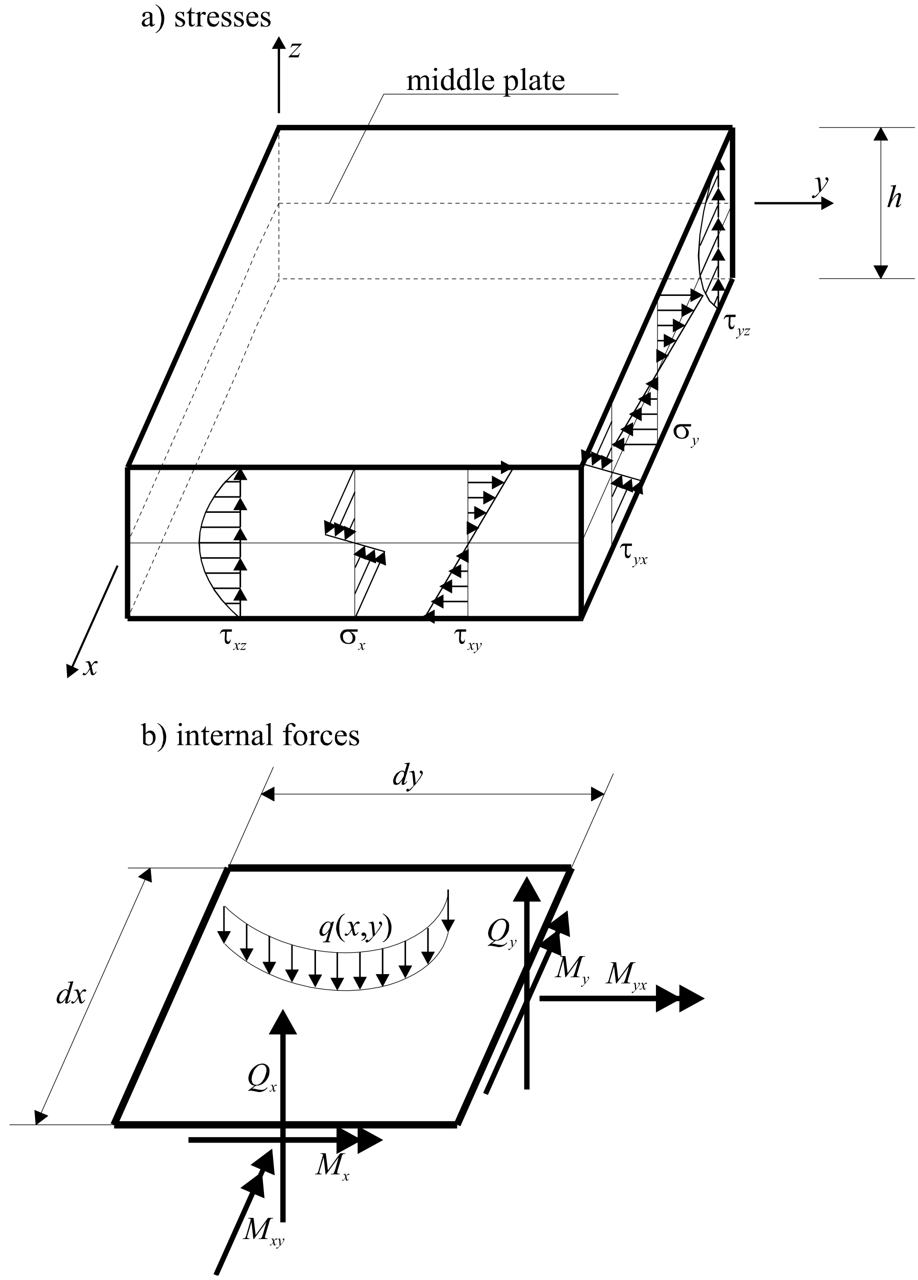

Fig.7.2

The equilibrium of an infinitesimal plate element shown in Fig.7.2b leads to the set of equations:

$$
\begin{aligned}
&\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q(x,y)=0\,,\\
&\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}=Q_x\,,\\
&\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}=Q_y\,.
\end{aligned}
\qquad (7.6)
$$

After doing integration (7.5) taking into consideration (7.4), we obtain

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right),$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right), \tag{7.7}$$

$$M_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y},$$

where D denotes the plate stiffness defined by the equation

$$D = \frac{Eh^3}{12(1-\nu^2)}. \tag{7.8}$$

From the last two equations (7.6) we obtain relations determining the shearing forces:

$$Q_x = -D\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right),$$

$$Q_y = -D\left(\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3}\right). \tag{7.9}$$

Inserting the above equation describing shearing forces into the first equations (7.6) we obtain

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x,y)}{D}. \tag{7.10}$$

It is a biharmonic partial differential equation which should be satisfied by the function of deflection $w(x,y)$ within the plate. The following boundary conditions should be realised at the edges of the plate:

a) $w = 0$, $\frac{\partial w}{\partial n} = 0$ - on the fixed edge,

b) $w = 0$, $\frac{\partial^2 w}{\partial n^2} = 0$ - on the free supported edge,

c) $M_n = 0$, $V_n = 0$ - on the free edge.

In the above equations n defines the direction of the line which is perpendicular to the edge and V_n is the reduced force led in by Kirchhoff [18] in 1850. This force joins the influence of the torsion moment M_{ns} and the shearing force Q_n on the free edge Fig.7.2b:

$$V_n = Q_n - \frac{\partial M_{ns}}{\partial s} = -D\left[\frac{\partial^3 w}{\partial n^3} + (2-\nu)\frac{\partial^3 w}{\partial n \partial s^2}\right] \tag{7.11}$$

where n describes the direction of the line which is perpendicular to the edge and s is the direction of the line which is parallel to the edge of the plate.

The modification of the boundary conditions is necessary here because the fourth order equation (7.10) cannot be solved for three boundary conditions coming from the requirement of stress disappearance on the free edge:
$M_{ns} = 0$, $M_n = 0$, $Q_n = 0$.

7.2. A FINITE TRIANGULAR ELEMENT OF A THIN PLATE

Now we show the way of building the stiffness matrix of a triangular element of a thin plate (Fig.7.3).

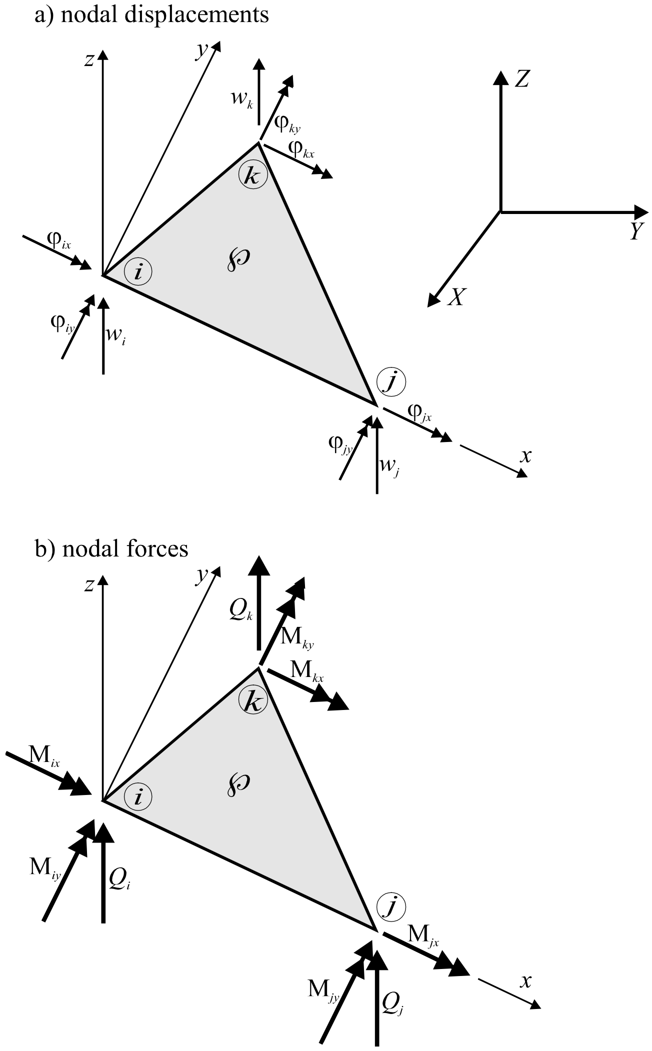

Fig.7.3

We also introduce a few convenient notations:

- $w(x,y)$ stands for the function of displacement of the middle plate of an element;

– $\varphi_x = \dfrac{\partial w}{\partial y}$ is the rotation angle of the element around the x axis;

– $\varphi_y = -\dfrac{\partial w}{\partial x}$ is a rotation angle of the element around the y axis.

As it is seen in Fig.7.3 the node of a plate element has three degrees of freedom. Hence nodal displacement vectors of the element in the local system can be written as follows:

$$\mathbf{u'}_i = \begin{bmatrix} w_i \\ \varphi_{ix} \\ \varphi_{iy} \end{bmatrix}, \ \mathbf{u'}_j = \begin{bmatrix} w_j \\ \varphi_{jx} \\ \varphi_{jy} \end{bmatrix}, \ \mathbf{u'}_k = \begin{bmatrix} w_k \\ \varphi_{kx} \\ \varphi_{ky} \end{bmatrix} \tag{7.12}$$

and an element displacement vector:

$$\mathbf{u'}^e = \begin{bmatrix} \mathbf{u'}_i \\ \mathbf{u'}_j \\ \mathbf{u'}_k \end{bmatrix}. \tag{7.13}$$

Directions of both nodal displacements and forces (Fig.7.3b) are the same, so the nodal forces vectors have similar notations:

$$\mathbf{f'}_i = \begin{bmatrix} Q_i \\ M_{ix} \\ M_{iy} \end{bmatrix}, \ \mathbf{f'}_j = \begin{bmatrix} Q_j \\ M_{jx} \\ M_{jy} \end{bmatrix}, \ \mathbf{f'}_k = \begin{bmatrix} Q_k \\ M_{kx} \\ M_{ky} \end{bmatrix}. \tag{7.14}$$

Hence we write the nodal force vector of the element as follows:

$$\mathbf{f'}^e = \begin{bmatrix} \mathbf{f'}_i \\ \mathbf{f'}_j \\ \mathbf{f'}_k \end{bmatrix}. \tag{7.15}$$

We approximate the surface of the deformed element by the polynomial of the third order proposed by J.LTocher in 1962:

$$w(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^3 + a_8\left(x^2 y + xy^2\right) + a_9 y^3 = \tag{7.16}$$

$$= \boldsymbol{\eta}^{\mathrm{T}}\mathbf{a},$$

where

$$
\boldsymbol{\eta}=\begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \\ x^3 \\ x^2y+xy^2 \\ y^3 \end{bmatrix}, \qquad \mathbf{a}=\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix}.
$$

We determine the coefficients $a_1 \dots a_9$ of the function $w(x,y)$ from the boundary conditions at the nodes i, j, k:

$$
\begin{aligned}
&w(x_i, y_i) = w_i,\ \varphi_x(x_i, y_i) = \varphi_{ix},\ \varphi_y(x_i, y_i) = \varphi_{iy}, \\
&w(x_j, y_j) = w_j,\ \varphi_x(x_j, y_j) = \varphi_{jx},\ \varphi_y(x_j, y_j) = \varphi_{jy}, \\
&w(x_k, y_k) = w_k,\ \varphi_x(x_k, y_k) = \varphi_{kx},\ \varphi_y(x_k, y_k) = \varphi_{ky}.
\end{aligned} \tag{7.17}
$$

After calculating the rotation angles, we obtain

$$
\begin{aligned}
&\varphi_x = \frac{\partial w(x,y)}{\partial y} = a_3 + a_5 x + 2a_6 y + a_8(x^2 + 2xy) + 3a_9 y^2, \\
&\varphi_y = -\frac{\partial w(x,y)}{\partial x} = -\left[a_2 + 2a_4 x + a_5 y + 3a_7 x^2 + a_8(2xy + y^2)\right].
\end{aligned} \tag{7.18}
$$

Now we insert equations (7.16) and (7.18) into boundary conditions (7.17) obtaining:

$$
\mathbf{M}\,\mathbf{a} = \mathbf{u}'^{e}, \tag{7.19}
$$

where **M** is the quadratic matrix dependent on nodal coordinates of the element.

$$
\mathbf{M}=\begin{array}{c}
\begin{array}{ccccccccc} A_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & x_j & 0 & x_j^2 & 0 & 0 & x_j^3 & 0 & 0 \\
0 & 0 & 1 & 0 & x_j & 0 & 0 & x_j^2 & 0 \\
0 & -1 & 0 & -2x_j & 0 & 0 & -3x_j^2 & 0 & 0 \\
1 & x_k & y_k & x_k^2 & x_k y_k & y_k^2 & x_k^3 & x_k^2 y_k + x_k y_k^2 & y_k^3 \\
0 & 0 & 1 & 0 & x_k & 2y_k & 0 & x_k^2 + 2x_k y_k & 3y_k^2 \\
0 & -1 & 0 & -2x_k & -y_k & 0 & -3x_k^2 & -2x_k y_k - y_k^2 & 0
\end{bmatrix}
\end{array}
\begin{array}{l} w_i \\ \varphi_{ix} \\ \varphi_{iy} \\ w_j \\ \varphi_{jx} \\ \varphi_{jy} \\ w_k \\ \varphi_{kx} \\ \varphi_{ky} \end{array} \tag{7.20}
$$

We can present the solution of equation (7.19) as follows:

$$\mathbf{a} = \mathbf{M}^{-1}\,\mathbf{u}'^{e}, \tag{7.21}$$

where $\mathbf{M}^{-1}$ is the inverse matrix of $\mathbf{M}$. The discovery of $\mathbf{M}^{-1}$ is possible when $\det\mathbf{M} \neq 0$ (comp. Appendix 1) which is not always the case in our problem because

$$\det\mathbf{M} = x_j^5 y_k^5 \left(2x_k + y_k - x_j\right) \tag{7.22}$$

It means that in cases when the node k of the element is on the line described by equation $y = x_j - 2x$, then the matrix $\mathbf{M}$ is singular. Thus, the problem is solved by changing the local coordinate system.

Now we calculate a strain vector determined by equation (7.3)

$$\boldsymbol{\varepsilon} = -z\,\partial w(x,y) = -z\,\partial\,\boldsymbol{\eta}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{u}'^{e} = -z\mathbf{B}^{*}\mathbf{M}^{-1}\mathbf{u}'^{e}, \tag{7.23}$$

where $\mathbf{B}^{*}=\partial\,\boldsymbol{\eta}^{\mathrm{T}}$ is a rectangular matrix of which components are equal to:

$$\mathbf{B}^{*} = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 6x & 2y & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2x & 6y \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4(x+y) & 0 \end{bmatrix}. \tag{7.24}$$

Comparising equation (7.23) with the definition of the geometric matrix $\mathbf{B}^{e}$ described by equations **Błąd! Nie można odnaleźć źródła odwołania.**) and (1.38) we obtain

$$\mathbf{B}^{e} = -z\,\mathbf{B}^{*}\mathbf{M}^{-1}. \tag{7.25}$$

Hence we can make use of the definition of the stiffness matrix contained in equation (1.50):

$$\begin{aligned} \mathbf{K}'^{e} &= \int_{V}\left(\mathbf{B}^{e}\right)^{\mathrm{T}}\mathbf{D}\mathbf{B}^{e}dV = \left(\mathbf{M}^{-1}\right)^{\mathrm{T}}\int_{-h/2}^{h/2} z^2 dz \int_{A}\left(\mathbf{B}^{*}\right)^{\mathrm{T}}\mathbf{D}\mathbf{B}^{*}dA\ \mathbf{M}^{-1} = \\ &= \frac{Eh^3}{12\left(1-\nu^2\right)}\left(\mathbf{M}^{-1}\right)^{\mathrm{T}}\int_{A}\left(\mathbf{B}^{*}\right)^{\mathrm{T}}\mathbf{D}\mathbf{B}^{*}dA\ \mathbf{M}^{-1}. \end{aligned} \tag{7.26}$$

After denoting the integration existing in the above equation by $\mathbf{K}^{*}$ and applying the definition of plate stiffness we have

$$\mathbf{K}'^{e} = D\left(\mathbf{M}^{-1}\right)^{\mathrm{T}}\mathbf{K}^{*}\mathbf{M}^{-1}. \tag{7.27}$$

After calculating the matrix multiplication inside the integration in equation (7.26), we have

$$\mathbf{K}^{*} = \frac{E}{1-\nu^2}\int_{A}\mathbf{S}\,dA\ , \tag{7.28}$$

where

$$\mathbf{S}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 4\nu & 12x & 4(y+x\nu) & 12y\nu \\ 0 & 0 & 0 & 0 & 2(1-\nu) & 0 & 0 & 4(x+y)(1-\nu) & 0 \\ 0 & 0 & 0 & 4\nu & 0 & 4 & 12x\nu & 4(y\nu+x) & 12y \\ 0 & 0 & 0 & 12x & 0 & 12x\nu & 36x^2 & 12\left(xy+x^2\nu\right) & 36xy\nu \\ 0 & 0 & 0 & 4(y+x\nu) & 4(x+y)(1-\nu) & 4(y\nu+x) & 12\left(xy+x^2\nu\right) & \begin{matrix} 4\left(3y^2+4xy+3x^2\right)+ \\ -8\nu\left(x^2+xy+y^2\right) \end{matrix} & 12\left(y^2\nu+xy\right) \\ 0 & 0 & 0 & 12y\nu & 0 & 12y & 36xy\nu & 12\left(y^2\nu+xy\right) & 36y^2 \end{bmatrix}.$$

While calculating the integration of functions exiting in equation (7.28), the following relations are helpful:

$$\begin{aligned} \int_A dA &= \frac{1}{2}x_j y_k, \\ \int_A x\,dA &= \frac{1}{6}x_j y_k\left(x_j+x_k\right), \\ \int_A y\,dA &= \frac{1}{6}x_j y_k^2, \\ \int_A x^2 dA &= \frac{1}{12}x_j y_k\left(x_j^2+x_j x_k+x_k^2\right), \\ \int_A xy\,dA &= \frac{1}{24}x_j y_k^2\left(x_j+2x_k\right), \\ \int_A y^2 dA &= \frac{1}{12}x_j y_k^2. \end{aligned} \tag{7.29}$$

Matrix (7.26) is determined in the local coordinate system. We have to transform it to the global coordinate system in accordance with relation (1.53):

$$\mathbf{K}^e=\mathbf{R}^e\mathbf{K}'^e\left(\mathbf{R}^e\right)^{\mathrm{T}}.$$

The rotation matrix of an element $\mathbf{R}^e$ is equal to:

$$\mathbf{R}^e=\begin{bmatrix} \mathbf{R}_i & & \\ & \mathbf{R}_j & \\ & & \mathbf{R}_k \end{bmatrix}, \tag{7.30}$$

where $\mathbf{R}_i$, $\mathbf{R}_j$, $\mathbf{R}_k$ are the transformation matrices of nodes. If we use the same coordinate systems for all the nodes (it has been done so in this chapter), then we can use only one transformation matrix: $\mathbf{R}_j=\mathbf{R}_i$, $\mathbf{R}_k=\mathbf{R}_i$,

$$\mathbf{R}_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}, \tag{7.31}$$

where $c = \cos\alpha$, $s = \sin\alpha$ and α is the angle between the X axis of the global system and the x axis of the local system (Fig.7.4). Value 1 in the first row of the matrix $\mathbf{R}_i$ is the consequence of a fact that axes Z and z are parallel.

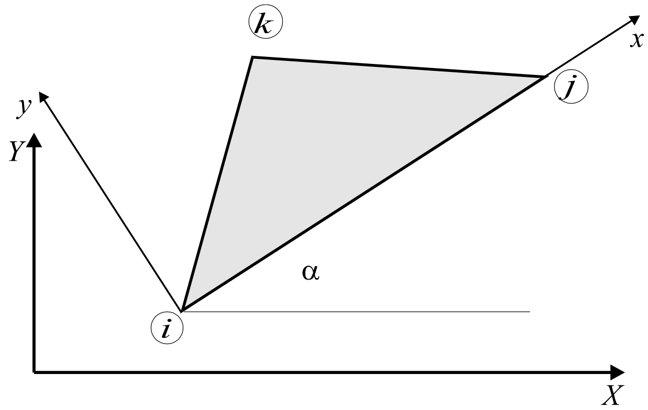

Fig.7.4

The triangular element of which the matrix stiffness has been obtained has a convenient feature. Namely, it allows to discrete plates of any shape without any difficulty. This element joined with a 2D triangular element can be used as a shell element (comp. [12]).

Elements of any other shapes (rectangular or quadrangular) are presented in books written by Bathe [1], Zienkiewicz [19],[20], Rao [13] or Rakowski [12].