CHAPTER IV.
STATICS OF A 2D FRAME SYSTEM

The right selection of a calculating model for a construction is very important for quality and exactness of obtained results. A decision if a given structure is a frame or a truss (for example, a truss with fixed nodes) is often subjective and it depends on experience and intuition of the person making calculations.

In this chapter we will present the following model of a bar structure - a plate frame which gives more possibilities of modelling real bar structures. The element of a 2D frame is more general than a truss element presented in Chapter II because with help of this element we can also model ideal truss structures (articulated connection of elements at nodes). We can simply say that a frame is a structure whose bars can be bended while truss elements can be only compressed and stretched. It has the following consequences:

- bar (an element) of a frame can be loaded between nodes,
- modelling of different types of loads is possible, for example: concentrated forces, concentrated moments, distributed loads,
- connection of an element with a node can be a fixed connection provided that the rotation angle of a node and of a nodal section of the element are identical or it can be an articulated connection when independent rotations of a node and a nodal section are possible,
- node of a plate frame has three degrees of freedom which means that we have to know two components of a translation vector: u_X, u_Y and the rotation angle φ_Z in order to determine the location of this node.

In case of plate frames we will neglect index Z of rotation angles in our notations because all rotation angles on the plate XY which we will use to describe the construction are rotation angles with respect to the Z axis. Let us assume that a frame element is straight and homogeneous which means that it is made from a homogeneous material and has a constant cross section. The view of a frame element, directions, senses of nodal displacements and forces which we will consider as positive are presented in Fig.4.1.

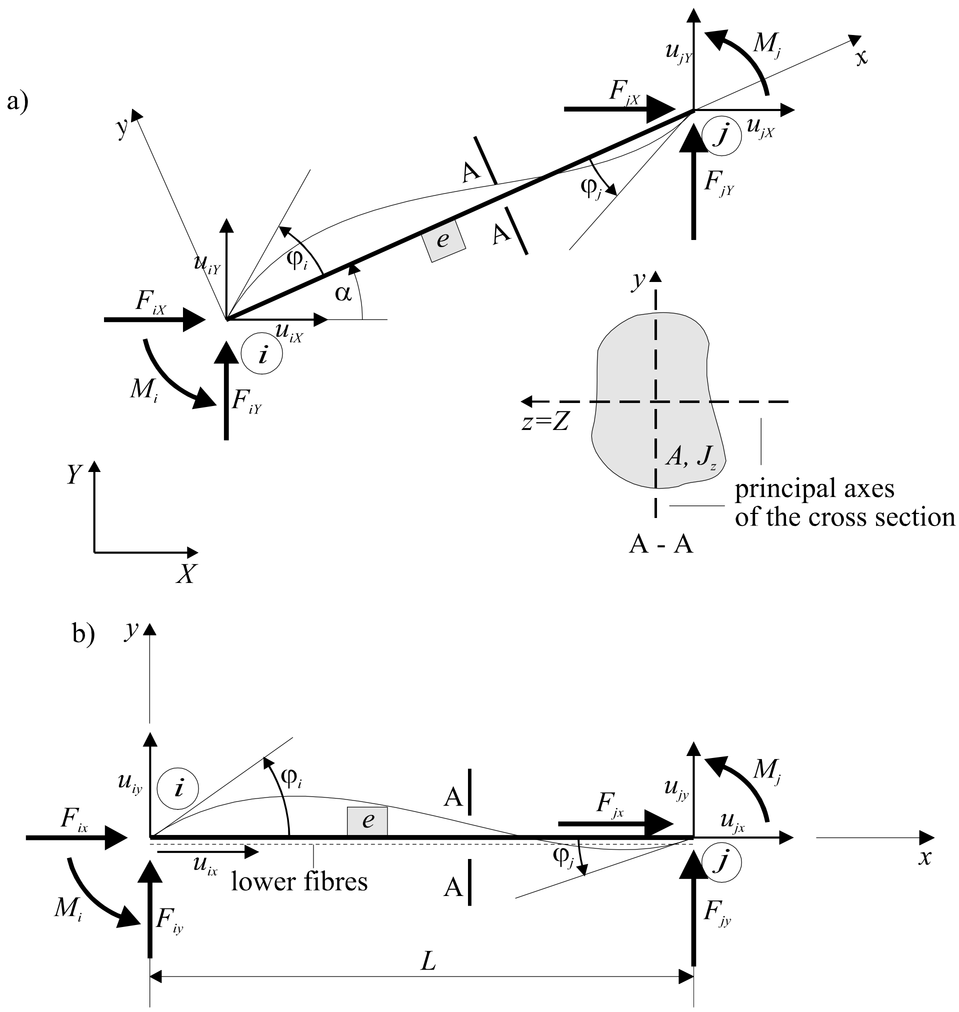

Fig.4.1

4.1. THE ELEMENT STIFNESS MATRIX FOR A PLATE FRAME

We group nodal displacements and forces shown in Fig.4.1a,b in column matrices just as we did it previously in Chapters II, III. They are called vectors:

– displacement vector of the first node *i* and the last node *j* in the local system (Fig.4.1b)

$$\mathbf{u'}_i = \begin{bmatrix} u_{ix} \\ u_{iy} \\ \varphi_i \end{bmatrix}, \ \mathbf{u'}_j = \begin{bmatrix} u_{jx} \\ u_{jy} \\ \varphi_j \end{bmatrix}. \tag{4.1}$$

– nodal forces vector in the local coordinate system

$$\mathbf{f'}_i = \begin{bmatrix} F_{ix} \\ F_{iy} \\ M_i \end{bmatrix}, \ \mathbf{f'}_j = \begin{bmatrix} F_{jx} \\ F_{jy} \\ M_j \end{bmatrix}. \tag{4.2}$$

– element displacement vector in the local coordinate system

$$\mathbf{u}'^{e} = \begin{bmatrix} \mathbf{u}'_i \\ \mathbf{u}'_j \end{bmatrix} = \begin{bmatrix} u_{ix} \\ u_{iy} \\ \varphi_i \\ u_{jx} \\ u_{jy} \\ \varphi_j \end{bmatrix}. \tag{4.3}$$

– element forces vector in the local coordinate system

$$\mathbf{f}'^{e} = \begin{bmatrix} \mathbf{f}'_i \\ \mathbf{f}'_j \end{bmatrix} = \begin{bmatrix} F_{ix} \\ F_{iy} \\ M_i \\ F_{jx} \\ F_{jy} \\ M_j \end{bmatrix}. \tag{4.4}$$

We can also describe all the formulated above vectors in the global system:

$$\mathbf{u}_i = \begin{bmatrix} u_{iX} \\ u_{iY} \\ \varphi_i \end{bmatrix}, \ \mathbf{u}_j = \begin{bmatrix} u_{jX} \\ u_{jY} \\ \varphi_j \end{bmatrix}, \tag{4.5}$$

$$\mathbf{f}_i = \begin{bmatrix} F_{iX} \\ F_{iY} \\ M_i \end{bmatrix}, \ \mathbf{f}_j = \begin{bmatrix} F_{jX} \\ F_{jY} \\ M_j \end{bmatrix}, \tag{4.6}$$

$$\mathbf{u}^{e} = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \end{bmatrix} = \begin{bmatrix} u_{iX} \\ u_{iY} \\ \varphi_i \\ u_{jX} \\ u_{jY} \\ \varphi_j \end{bmatrix}, \tag{4.7}$$

$$\mathbf{f}^{e} = \begin{bmatrix} \mathbf{f}_i \\ \mathbf{f}_j \end{bmatrix} = \begin{bmatrix} F_{iX} \\ F_{iY} \\ M_i \\ F_{jX} \\ F_{jY} \\ M_j \end{bmatrix}. \tag{4.8}$$

As in the previous chapters the relation between nodal forces and nodal displacements will be of great importance. This relation (analogous to equation **Błąd! Nie można odnaleźć źródła odwołania.**) for a truss) in the local coordinate system has the form:

$$\mathbf{K}'^{e}\,\mathbf{u}'^{e} = \mathbf{f}'^{e}, \tag{4.9}$$

and in the global system

$$\mathbf{K}^{e}\,\mathbf{u}^{e} = \mathbf{f}^{e}. \tag{4.10}$$

At this moment we will concentrate on searching for the stiffness matrix $\mathbf{K}'^{e}$ in the local coordinate system and next its transformation to the global system.

Equilibrium equations of the element presented in Fig.4.1b lead to the following relations between nodal forces:

$$\begin{aligned}
&\sum F_x = F_{ix} + F_{jx} = 0 \quad \rightarrow \quad F_{ix} = -F_{jx};\\
&\sum F_y = F_{iy} + F_{jy} = 0;\\
&\sum M_i = M_i + M_j + F_{jy}\,L = 0.
\end{aligned} \tag{4.11}$$

As it has been shown, three equations are still lacking to calculate six components of the vector $\mathbf{f}'^{e}$. The discussion concerning element strains will give these equations. The deformation caused by the axial forces F_{ix} and F_{jx} is identical to the deformation of a truss element, hence we take the advantage of previously determined dependence (2.11),(2.12a). We will obtain the remaining equations when we consider the flexural deformation of an element and the relation between shearing forces and bending moments. As it is known the following relation between a curvature and a bending moment is (comp. [8]):

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} = \frac{M(x)}{EJ_z}, \tag{4.12}$$

where ρ represents the radius of a curvature, E is Young's modulus of a material, J_z is the moment of inertia of an element cross section (comp. Fig.4.1).

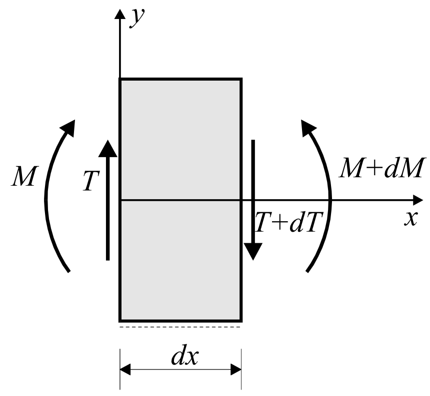

Fig.4.2

The equilibrium of one part of a bending bar (Fig.4.2) gives the equation:

$$T(x) = \frac{dM(x)}{dx}. \tag{4.13}$$

The opposite sign of the right hand side of the equation (4.12) (comp. [8]) to the one that we have usually assumed, comes from the sense of the y axis of the local coordinate system which is orientated up in our assumptions.

Since we will deal with linear structures with small deflections, we assume $\frac{dy}{dx} << 1$, which allows to simplify equation (4.12) to the known form:

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EJ_z}. \tag{4.14}$$

Differenting this equation twice we obtain the relation (comp. [8], [10]):

$$\frac{d^4y}{dx^4} = \frac{q_y(x)}{EJ_z}, \tag{4.15}$$

where $q_y(x)$ denotes the distributed load which is perpendicular to the axis of an element. Here the considered element is free from nodal loads, thus, $q_y \equiv 0$.

Finally, we obtain the searched set of differential equations:

a) $\frac{d^4y}{dx^4} = 0$,

b) $\frac{d^2y}{dx^2} = \frac{M(x)}{EJ_z}$, (4.16)

c) $\frac{d^3y}{dx^3} = \frac{T(x)}{EJ_z}$.

After integrating relations (4.16a) we obtain equations:

- bending line of the frame element:

$$y(x) = C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4 , \tag{4.17}$$

- bending moment:

$$M(x) = EJ_z \left[C_1 x + C_2 \right], \tag{4.18}$$

- shearing force:

$$T(x) = EJ_z C_1 , \tag{4.19}$$

where C_1 ... C_4 stand for integration constants which should be determined on the basis of boundary conditions.

We have four boundary conditions:

- at node i , x=0:

$$\begin{aligned} & y(0) = u_{iy} , \\ & \left.\frac{dy}{dx}\right|_{x=0} = \varphi_i , \end{aligned} \tag{4.20}$$

- at node j , x=L:

$$\begin{aligned} & y(L) = u_{jy} , \\ & \left.\frac{dy}{dx}\right|_{x=L} = \varphi_j . \end{aligned} \tag{4.21}$$

After inserting these conditions into equation (4.17), we obtain the following values of the integration constants:

$$\begin{aligned} & C_1 = \frac{6}{L^2}\left(\varphi_i + \varphi_j - 2\frac{u_{jy} - u_{iy}}{L} \right), \\ & C_2 = -\frac{1}{L}\left(4\varphi_i + 2\varphi_j - 6\frac{u_{jy} - u_{iy}}{L} \right), \\ & C_3 = \varphi_i , \\ & C_4 = u_{iy} . \end{aligned} \tag{4.22}$$

Hence after putting the above equations into equations (4.18), (4.19) and considering the senses of both nodal and bending moments (comp. Fig.4.1 and Fig.4.2), we obtain

$$M_i = -M(0) = \frac{EJ_z}{L}\left[4\varphi_i + 2\varphi_j - 6\frac{u_{jy} - u_{iy}}{L} \right],$$

$$M_j = M(L) = \frac{EJ_z}{L}\left[2\varphi_i + 4\varphi_j - 6\frac{u_{jy} - u_{iy}}{L}\right],$$

$$F_{iy} = T(0) = \frac{EJ_z}{L^2}\left[6\varphi_i + 6\varphi_j - 12\frac{u_{jy} - u_{iy}}{L}\right],$$

$$F_{jy} = -T(L) = \frac{EJ_z}{L^2}\left[-6\varphi_i - 6\varphi_j + 12\frac{u_{jy} - u_{iy}}{L}\right]. \tag{4.23}$$

Finally, tabulating equations (2.12a) and (4.23) in a suitable sequence we obtain the searched stiffness matrix:

$$\mathbf{K}'^e = \begin{bmatrix} \frac{EA}{L} & & & -\frac{EA}{L} & & \\ & 12\frac{EJ_z}{L^3} & 6\frac{EJ_z}{L^2} & & -12\frac{EJ_z}{L^3} & 6\frac{EJ_z}{L^2} \\ & 6\frac{EJ_z}{L^2} & 4\frac{EJ_z}{L^2} & & -6\frac{EJ_z}{L^2} & 2\frac{EJ_z}{L^2} \\ -\frac{EA}{L} & & & \frac{EA}{L} & & \\ & -12\frac{EJ_z}{L^3} & -6\frac{EJ_z}{L^2} & & 12\frac{EJ_z}{L^3} & -6\frac{EJ_z}{L^2} \\ & 6\frac{EJ_z}{L^2} & 2\frac{EJ_z}{L^2} & & -6\frac{EJ_z}{L^2} & 4\frac{EJ_z}{L^2} \end{bmatrix}, \tag{4.24}$$

The relations described by equations (4.23) are called transformation formulae of the displacement method in structure mechanics (in some other form) (comp. [10]).

4.2. TRANSFORMATION OF THE SIFFNESS MATRIX FROM THE GLOBAL COORDINATE SYSEM TO THE LOCAL ONE

The transfer of the matrix $\mathbf{K}'^e$ to the global coordinate system is done according to rules analogous to the rules described by equation (2.34) in Sec.2.4. In order to make the transformation matrix of an element we need $\mathbf{R}_i$ that is, the transformation matrix from the local system to the global one for the node i. Since the third degree of freedom of frame nodes is a rotation with respect to the z axis which does not change its location because it is always perpendicular to the plane xy, thus, the rotation will be the same as for a truss element:

$u_{iX} = u_{ix}\cos\alpha - u_{iy}\sin\alpha$,

$u_{iY} = u_{ix} \sin\alpha + u_{iy} \cos\alpha$,

$\varphi_{iZ} = \varphi_{iz} = \varphi_i$,

or in the matrix form

$$\begin{bmatrix} u_{iX} \\ u_{iY} \\ \varphi_i \end{bmatrix} = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{ix} \\ u_{iy} \\ \varphi_i \end{bmatrix}, \ \mathbf{u}_i = \mathbf{R}_i \, \mathbf{u}'_i ,$$

$$\text{where } \mathbf{R}_i = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.25}$$

In accord with the assumption accepted in the introduction that the frame element is straight, the transformation matrix of the node j is identical to $\mathbf{R}_i$ which leads to the final form of the element stiffness matrix:

$$\mathbf{R}^e = \begin{bmatrix} c & -s & 0 & 0 & 0 & 0 \\ s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.26}$$

After multiplying matrices described by equation (2.34) we obtain the stiffness matrix of a frame element in the global coordinate system. Unfortunately, its form is rather complex:

$$\mathbf{K}^e = \frac{EJ_z}{L^2} \begin{array}{c} \begin{array}{cccccc} u_{ix} & u_{iy} & \varphi_{ix} & u_{jx} & u_{jy} & \varphi_{jx} \end{array} \\ \begin{bmatrix} \frac{1}{L}\left(\frac{c^2}{\lambda^2}+12s^2\right) & \frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & \text{-}6s & -\frac{1}{L}\left(\frac{c^2}{\lambda^2}+12s^2\right) & -\frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & \text{-}6s \\ \frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & \frac{1}{L}\left(\frac{s^2}{\lambda^2}+12c^2\right) & 6c & -\frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & -\frac{1}{L}\left(\frac{s^2}{\lambda^2}+12c^2\right) & 6c \\ \text{-}6s & 6c & 4L & \text{-}6s & \text{-}6c & 2L \\ -\frac{1}{L}\left(\frac{c^2}{\lambda^2}+12s^2\right) & -\frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & \text{-}6s & \frac{1}{L}\left(\frac{c^2}{\lambda^2}+12s^2\right) & \frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & 6s \\ -\frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & -\frac{1}{L}\left(\frac{s^2}{\lambda^2}+12c^2\right) & \text{-}6c & \frac{sc}{L}\left(\frac{1}{\lambda^2}-12\right) & \frac{1}{L}\left(\frac{s^2}{\lambda^2}+12c^2\right) & \text{-}6c \\ \text{-}6s & 6c & 2L & 6s & \text{-}6c & 4L \end{bmatrix} \end{array} \begin{array}{c} F_{ix} \\ F_{iy} \\ M_i \\ F_{jx} \\ F_{jy} \\ M_j \end{array} \tag{4.27}$$

$\lambda^2 = \dfrac{J_z}{AL^2}$ $\qquad c = \cos\alpha \qquad s = \sin\alpha$

4.3. STATIC CONDENSATION OF THE STIFFNESS MATRIX

A frame element is not always joined with a node in the way ensuring the agreement of all nodal displacements and displacements in the bar section at this node. Articulated joints shown in Fig.4.3 are most often such incomplete connections.

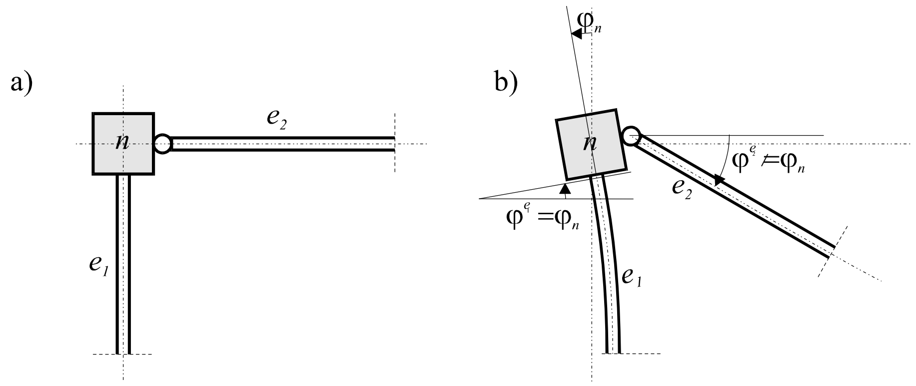

Fig.4.3

At this joint the angle of the nodal rotation does not influence the rotation of the element section of a node. The latter can rotate independently of the node (the element e_2 in Fig.4.3).

We determine the unknown angle of the rotation of such an element using an additional equation which is given by the equilibrium condition of moments in a joint. Hence we can reduce the number of degrees of freedom of the element because the additional equilibrium condition allows to eliminate one displacement from the set of equations. We show the way of eliminating the degree of freedom on the example of two types of connections of an element with a node.

Example No 1 - articulated connection (Fig.4.4).

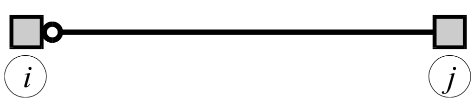

Fig.4.4

Additional equilibrium condition of a section at the node i :

$$M_i = 0, \tag{4.28}$$

leads after considering equations (4.2), (4.9) and (4.24) to conditions:

$$\frac{EJ_z}{L}\left[6\frac{u_{iy}}{L} + 4\varphi_i - 6\frac{u_{jy}}{L} + 2\varphi_j\right] = 0, \tag{4.29}$$

and thus we calculate the searched value of the rotation angle of the section at the node i:

$$\varphi_i = -\frac{3}{2}\frac{u_{iy}}{L} + \frac{3}{2}\frac{u_{jy}}{L} - \frac{1}{2}\varphi_j . \tag{4.30}$$

After putting this result into equation (4.9) and taking into consideration matrix (4.24) we obtain:

$$\begin{aligned}
F_{iy} &= \frac{EJ_z}{L^2}\left[12\frac{u_{iy}}{L} + 6\left(-\frac{3}{2}\frac{u_{iy}}{L} + \frac{3}{2}\frac{u_{jy}}{L} - \frac{1}{2}\varphi_j\right) - 12\frac{u_{jy}}{L} + 6\varphi_j\right] = \\
&= \frac{EJ_z}{L^2}\left[3\frac{u_{iy}}{L} - 3\frac{u_{jy}}{L} + 3\varphi_j\right], \\
F_{jy} &= \frac{EJ_z}{L^2}\left[-12\frac{u_{iy}}{L} - 6\left(-\frac{3}{2}\frac{u_{iy}}{L} + \frac{3}{2}\frac{u_{jy}}{L} - \frac{1}{2}\varphi_j\right) + 12\frac{u_{jy}}{L} - 6\varphi_j\right] = \\
&= \frac{EJ_z}{L^2}\left[-3\frac{u_{iy}}{L} + 3\frac{u_{jy}}{L} - 3\varphi_j\right], \\
M_j &= \frac{EJ_z}{L}\left[6\frac{u_{iy}}{L} + 2\left(-\frac{3}{2}\frac{u_{iy}}{L} + \frac{3}{2}\frac{u_{jy}}{L} - \frac{1}{2}\varphi_j\right) - 6\frac{u_{jy}}{L} + 4\varphi_j\right] = \\
&= \frac{EJ_z}{L}\left[3\frac{u_{iy}}{L} - 3\frac{u_{jy}}{L} + 3\varphi_j\right],
\end{aligned} \tag{4.31}$$

and hence the new stiffness matrix of an element with the joint at the node i:

$$\mathbf{K}'^{e(3,i)} = \left[\begin{array}{c:c:c:c:c:c}
\frac{EA}{L} & 0 & & -\frac{EA}{L} & 0 & 0 \\ \hdashline
0 & 3\frac{EJ_z}{L^3} & & 0 & -3\frac{EJ_z}{L^3} & 3\frac{EJ_z}{L^2} \\ \hdashline
 & & & & & \\ \hdashline
-\frac{EA}{L} & 0 & & \frac{EA}{L} & 0 & 0 \\ \hdashline
0 & -3\frac{EJ_z}{L^3} & & 0 & 3\frac{EJ_z}{L^3} & -3\frac{EJ_z}{L^2} \\ \hdashline
0 & 3\frac{EJ_z}{L^2} & & 0 & -3\frac{EJ_z}{L^2} & 3\frac{EJ_z}{L}
\end{array}\right]. \tag{4.32}$$

Upper indices $(3,i)$ in the notation of the stiffness matrix (4.32) inform that the third degree of freedom is eliminated at the first node.

Example No 2 - moveable connection(Fig.4.5)

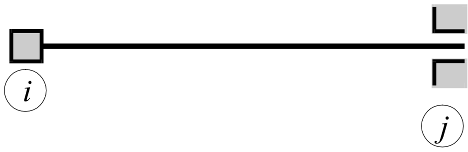

Fig.4.5

Here the additional condition is the disappearance of the axial force at the node j:

$$F_{jx} = 0, \tag{4.33}$$

which after analogous transformations leads to the equation:

$$F_{ix} = 0, \tag{4.34}$$

and it does not change relations for the remaining nodal forces.

The stiffness matrix of such an element takes the following form:

$$\mathbf{K}'^{e(1,j)} = \begin{bmatrix} & & & & & \\ & 12\frac{EJ_z}{L^3} & 6\frac{EJ_z}{L^2} & & -12\frac{EJ_z}{L^3} & 6\frac{EJ_z}{L^2} \\ & 6\frac{EJ_z}{L^2} & 4\frac{EJ_z}{L^2} & & -6\frac{EJ_z}{L^2} & 2\frac{EJ_z}{L^2} \\ & & & & & \\ & -12\frac{EJ_z}{L^3} & -6\frac{EJ_z}{L^2} & & 12\frac{EJ_z}{L^3} & -6\frac{EJ_z}{L^2} \\ & 6\frac{EJ_z}{L^2} & 2\frac{EJ_z}{L^2} & & -6\frac{EJ_z}{L^2} & 4\frac{EJ_z}{L^2} \end{bmatrix}. \tag{4.35}$$

Upper indices $(1,j)$ in the notation of this matrix inform that the first degree of freedom at the last node of an element has been eliminated.

The presented above process is called the static condensation of a stiffness matrix. Now we will give the matrix notation of an operation leading to a condensed stiffness matrix. For the sake of simplicity we assume the last degree of freedom of an element is the eliminated degree of freedom. Nodal forces, nodal displacements vectors and the stiffness matrix are divided into blocks:

$$\begin{bmatrix} \mathbf{f}_1 \\ \hline \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{10} \\ \hline & \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \end{bmatrix}, \tag{4.36}$$

$$\begin{bmatrix} \mathbf{f}_0 \end{bmatrix} \begin{bmatrix} \mathbf{K}_{01} & \mathbf{K}_{00} \end{bmatrix} \begin{bmatrix} \mathbf{u}_0 \end{bmatrix}$$

where according to the symmetry of the matrix we have

$\mathbf{K}_{11} = \mathbf{K}_{11}^{\mathrm{T}}$, $\mathbf{K}_{01} = \mathbf{K}_{10}^{\mathrm{T}}$,

$\mathbf{K}_{00}$ is the matrix with dimension 1x1 and thus it is a scalar, the blocks $\mathbf{f}_0$ and $\mathbf{u}_0$ are scalars, too. The results of the multiplication of matrix blocks (4.36) are:

a) $\mathbf{f}_1 = \mathbf{K}_{11}\,\mathbf{u}_1 + \mathbf{K}_{10}\,\mathbf{u}_0$,

b) $\mathbf{f}_0 = 0 = \mathbf{K}_{01}\,\mathbf{u}_1 + \mathbf{K}_{00}\,\mathbf{u}_0$ - scalar. (4.37)

From equation (4.37b) we calculate

$$\mathbf{u}_0 = -\mathbf{K}_{00}^{-1}\,\mathbf{K}_{01}\,\mathbf{u}_1, \tag{4.38}$$

and after inserting this relation into (4.37a) we obtain

$$\mathbf{f}_1 = \mathbf{K}_{11}\mathbf{u}_1 - \mathbf{K}_{10}\mathbf{K}_{00}^{-1}\mathbf{K}_{01}\mathbf{u}_1, \tag{4.39}$$

or less

$$\mathbf{f}_1 = \mathbf{K}''\mathbf{u}_1, \tag{4.40}$$

where

$$\mathbf{K}'' = \mathbf{K}_{11} - \mathbf{K}_{10}\mathbf{K}_{00}^{-1}\mathbf{K}_{10}^{\mathrm{T}}, \tag{4.41}$$

is the condensed element stiffness matrix.

Vector $\mathbf{f}_1$ of an element load still remains to be determined. We obtain it by composing both the load vector $\mathbf{f}^{o}$ of an element with rigid connections with nodes and the vector $\mathbf{f}^{u}$ of the load caused by displacements of nodes exempting from constraints

$$\mathbf{f} = \mathbf{f}^{o} - \mathbf{f}^{u} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1^{o} \\ \mathbf{f}_0^{o} \end{bmatrix} - \begin{bmatrix} \mathbf{f}_1^{u} \\ \mathbf{f}_0^{u} \end{bmatrix}. \tag{4.42}$$

Since

$$\mathbf{f}_0 = \mathbf{f}_0^{o} - \mathbf{f}_0^{u} = 0, \tag{4.43}$$

then

$$\mathbf{f}_0^{u} = \mathbf{f}_0^{o} = \mathbf{K}_{01}^{o}\mathbf{u}_1 + \mathbf{K}_{00}^{o}\mathbf{u}_0, \tag{4.44}$$

and hence

$$\mathbf{u}_0 = \left(\mathbf{K}_{00}^{o}\right)^{-1}\mathbf{f}_0^{o}, \tag{4.45}$$

because other displacements contained in $\mathbf{u}_1$ are equal to zero. Finally, we obtain

$$\mathbf{f}_1 = \mathbf{f}_1^{o} - \mathbf{K}_{10}^{o}\left(\mathbf{K}_{00}^{o}\right)^{-1}\mathbf{f}_0^{o}. \tag{4.46}$$

In the presented above way we can eliminate any degree of freedom but it requires some more complex transformations. We leave this problem to be solved by the reader.

4.4. BOUNDARY CONDITIONS OF PLANE FRAME STRUCTURES

Supports of plane frames include articulated and fixed supports all listed in Chapter II. The latter ones prevent the rotation of a support node. Symbolic notation of these supports and the boundary conditions describing them are shown in Fig.4.6

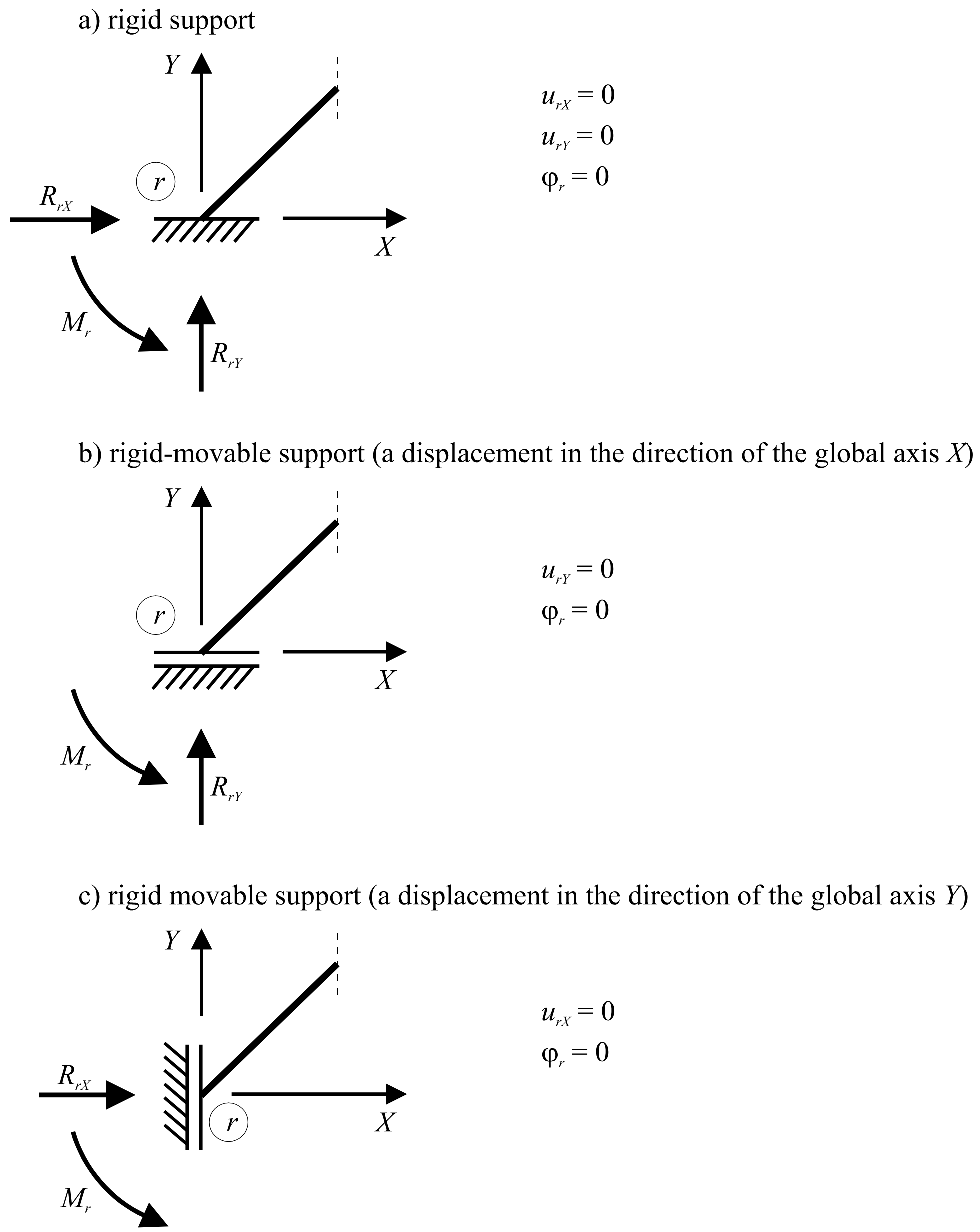

Fig.4.6

Considering boundary conditions requires the modification of a global stiffness matrix of a structure and it is done identically as for a plane truss (Sec.2.6), thus, we will not describe the way of modifying this matrix here. A whole range of other supports such as moveable skew supports and elastic supports considered analogously to supports of trusses described in Chapter II is also possible.

As a general method of consideration of non-typical supports we propose to consider the use of suitable boundary elements instead of these supports. We will discuss it in the next section.

4.5. BOUNDARY ELEMENTS OF 2D FRAMES

Introducing a boundary element is a convenient way to avoid problems connected with the consideration of different, non-typical boundary conditions. It allows, in fact, to model fixed and fixed-movable supports with approximate exactness and to substitute elastic supports.

Now we will present a single elastic support slanted at some angle. The scheme of this element and used notations are shown in Fig.4.7.

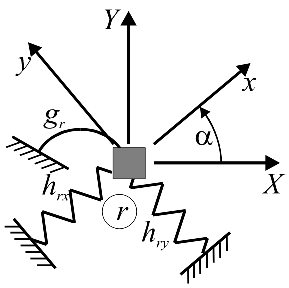

Fig.4.7

Stiffness of springs: h_{rx} and h_{ry} are forces which should be applied to their ends in order to induce their unitary extensions. Rotation stiffness of a support g_r is a moment necessary to induce the rotation of the node r equal to a unit angle.

The stiffness matrix of such an element in the local coordinate system has the form:

$$\mathbf{K}^b = \begin{bmatrix} h_{rx} & 0 & 0 \\ 0 & h_{ry} & 0 \\ 0 & 0 & g_r \end{bmatrix}. \tag{4.47}$$

Its transformation to the global system is done analogously to the case of normal frame or truss elements except that it concerns one node only (2.34). The rotation matrix is written by equation (4.15). Hence we can write the equation transforming the matrix $\mathbf{K}'^b$ to the global system:

$$\mathbf{K}^b = \mathbf{R}_r \, \mathbf{K}'^b \, \mathbf{R}_r^{\mathrm{T}} . \tag{4.48}$$

After taking into consideration equations (4.15) and (4.47)we obtain

$$\mathbf{K}^b = \begin{bmatrix} c^2 h_{rx} + s^2 h_{ry} & sc\left(h_{rx} - h_{ry}\right) & 0 \\ sc\left(h_{rx} - h_{ry}\right) & c^2 h_{rx} + s^2 h_{ry} & 0 \\ 0 & 0 & g_r \end{bmatrix}, \tag{4.49}$$

where $s = \sin\alpha$, $c = \cos\alpha$.

If we model flexible supports we ought to assume high stiffness of a suitable spring. In most cases stiffness of the order of $1 \cdot 10^{30}$ assures correspondence between results obtained with this method and the results obtained with the exact method.

4.6. INTERNAL FORCES DUE TO A STATIC LOAD

Variety of loads which can act on a frame structure is considerably greater than it was in the case of a truss. Frame elements can be affected by concentrated (forces, moments), distributed (pressure, moment loads) and temperature loads. The formulation of equilibrium equations requires substitution of internode loads for an equivalent set of concentrated forces and moments acting on nodes. The way of reduction of loads will be the subject of our discussion in this section.

Equations (4.17) and (4.22) define displacements of an element bending in the direction of the y axis of the global system. After adding the equations describing the displacements in an axial direction, we obtain relations defining the displacements vector for any point between nodes

$$\mathbf{u}(x) = \begin{bmatrix} u_x(x) \\ u_y(x) \\ \varphi(x) \end{bmatrix} = \mathbf{N}\,\mathbf{u}^e , \tag{4.50}$$

where **N** is the rectangular matrix of shape functions. It contains two blocks: $\mathbf{N}_i(x)$ - matrix of the shape functions for the first node and $\mathbf{N}_j(x)$ - matrix of the shape functions for the last node.

$$\mathbf{N}(x) = \begin{bmatrix} \mathbf{N}_i(x) & \mathbf{N}_j(x) \end{bmatrix}. \tag{4.51}$$

We can obtain both matrices from equations (4.17) and (2.10):

$$\mathbf{N}_i(x)=\begin{bmatrix}\omega_1(\xi) & 0 & 0\\ 0 & \omega_3(\xi) & L\omega_5(\xi)\\ 0 & \frac{1}{L}\omega_3'(\xi) & \omega_5'(\xi)\end{bmatrix},$$

$$\mathbf{N}_j(x)=\begin{bmatrix}\omega_2(\xi) & 0 & 0\\ 0 & \omega_4(\xi) & L\omega_6(\xi)\\ 0 & \frac{1}{L}\omega_4'(\xi) & \omega_6'(\xi)\end{bmatrix}, \qquad (4.52)$$

where nondimensional displacement functions $\omega_i(\xi)\,(i = 1,2 \ldots 6))$ and their derivatives $\omega_i'(\xi)$, $\omega_i''(\xi)$ are juxtaposed in Tab.4.1. The convenient nondimensional coordinate $\xi = x / L$ is introduced here.

Tab.4.1

Let us consider now the bar (an element) of a plate frame loaded with static loads (Fig.4.8).

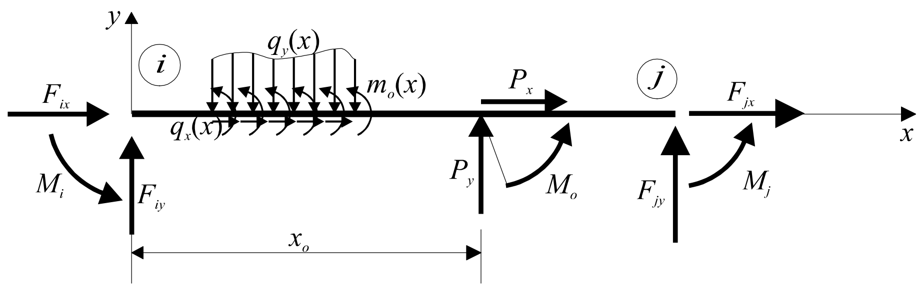

Fig.4.8

We will find nodal forces $\mathbf{f}^e$ by making use of conditions of element equilibrium. We will use the principle of virtual work here:

$$W_n = \left(\mathbf{f}'^e\right)^{\mathrm{T}}\mathbf{u}^e \qquad (4.53a)$$

where L_n is the work of nodal forces,

$$W_z = \int_0^L \left[q_y(x)\,u_y(x) + q_x(x)\,u_x(x) + m_o(x)\,\varphi(x)\right]dx \qquad (4.53b)$$

where L_z is the work of external forces (static loads).

Concentrated forces and moments can also be analysed by describing them in the following way:

$$q(x) = \delta(x - x_o)P\,,\ m_o = \delta(x - x_o)M_o\,, \qquad (4.54)$$

where $\delta(x_o)$ is Dirac's delta defined as follows (comp.[11]):

$$\begin{aligned}
&\delta(x-x_o)=0 \quad \text{for } x<x_o\,,\\
&\delta(x-x_o)\to\infty \quad \text{for } x=x_o\,,\\
&\delta(x-x_o)=0 \quad \text{for } x>x_o\,.
\end{aligned} \tag{4.55}$$

The element equilibrium is kept when

$W_n+W_z=0$, which means

$$\left(\mathbf{f}'^{e}\right)^{\mathrm{T}}\mathbf{u}^{e}=-\int_0^L\left[\mathbf{q}(x)\right]^{\mathrm{T}}\mathbf{u}(x)dx\,, \tag{4.56}$$

where $\mathbf{q}(x)$ is the vector of external loads:

$$\mathbf{q}(x)=\begin{bmatrix} q_x(x)\\ q_y(x)\\ m_o(x)\end{bmatrix}. \tag{4.57}$$

Putting the expression describing the element displacements vector (4.50) into (4.56) we obtain relations:

$$\left(\mathbf{f}'^{e}\right)^{\mathrm{T}}\mathbf{u}^{e}=-\int_0^L\mathbf{q}^{\mathrm{T}}\,\mathbf{N}\,\mathbf{u}^{e}\,dx\,, \tag{4.58}$$

$$\left(\mathbf{f}'^{e}\right)^{\mathrm{T}}=-\int_0^L\mathbf{N}\,\mathbf{q}\,dx\,, \tag{4.59}$$

which enable to replace loads acting on elements by loads acting on nodes. It should be noted here that there are forces acting on the nodes in the equilibrium equations and that these forces act against those acting on the element (comp. Fig. 2.11) thus, they should be subtracted from the nodal forces vector of the structure.

We check the effectiveness of equation (4.59) for three simple examples when:

1. the load of a concentrated force is applied to the center of an element,
2. the load of a concentrated moment,
3. the distributed load which is constant for the whole element.

Example No 1.

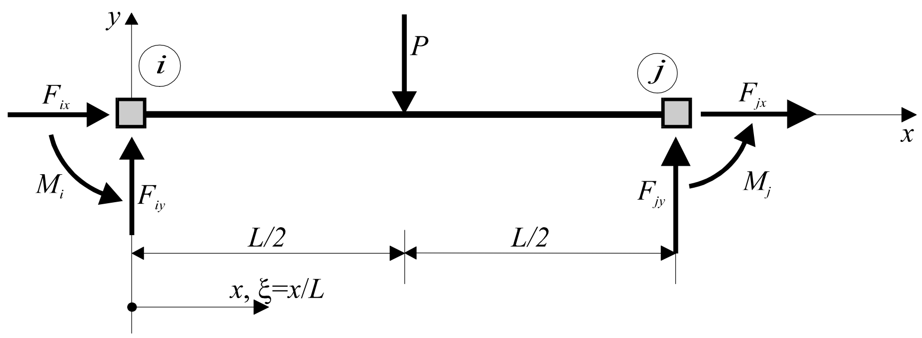

Fig.4.9

We introduce nondimensional coordinate $\xi = x / L$ to make the calculations more convenient and write the concentrated force as follows:

$$\mathbf{q}(\xi) = \begin{bmatrix} 0 \\ P\delta(\xi - 0.5) \\ 0 \end{bmatrix}$$

and after putting it into equation (4.59) we obtain

$$\mathbf{f}'^{e} = -\int_0^1 \begin{bmatrix} (\mathbf{N}_i)^{\mathrm{T}} \\ (\mathbf{N}_j)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} 0 \\ -P\delta(\xi - 0.5) \\ 0 \end{bmatrix} d\xi = P\int_0^1 \begin{bmatrix} 0 \\ \delta(\xi - 0.5)\omega_3(\xi) \\ L\delta(\xi - 0.5)\omega_5(\xi) \\ 0 \\ \delta(\xi - 0.5)\omega_4(\xi) \\ L\delta(\xi - 0.5)\omega_6(\xi) \end{bmatrix} d\xi =$$

$$= P\begin{bmatrix} 0 \\ \omega_3(0.5) \\ L\omega_5(0.5) \\ 0 \\ \omega_4(0.5) \\ L\omega_6(0.5) \end{bmatrix} = P\begin{bmatrix} 0 \\ 0.5 \\ L/8 \\ 0 \\ 0.5 \\ -L/8 \end{bmatrix}$$

which means that

$$F_{ix} = 0, \qquad F_{iy} = \frac{1}{2}P, \qquad M_i = \frac{1}{8}PL,$$

$$F_{jx} = 0, \qquad F_{jy} = \frac{1}{2}P, \qquad M_j = -\frac{1}{8}PL.$$

Example No 2.

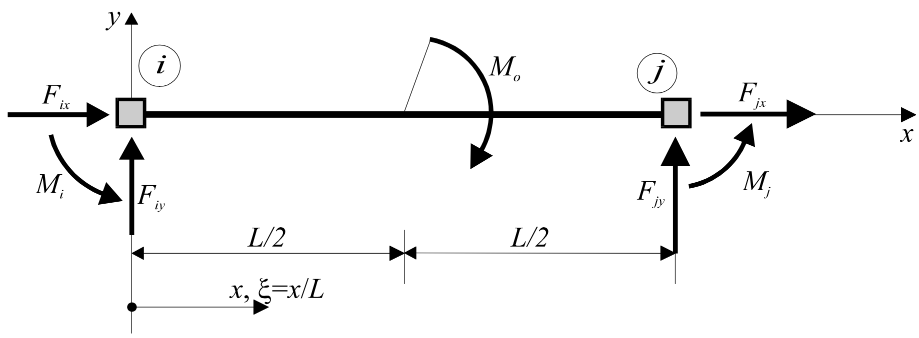

Fig.4.10

We write the concentrated moment applied to the center of an element by using Dirac's delta:

$$\mathbf{q}(\xi) = \begin{bmatrix} 0 \\ 0 \\ -M_o\delta(\xi-0.5) \end{bmatrix}$$

After inserting the load vector into equation (4.59), we obtain

$$\mathbf{f'}^{e} = M_o\int_0^1 \begin{bmatrix} (\mathbf{N}_i)^{\mathrm{T}} \\ (\mathbf{N}_j)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \delta(\xi-0.5) \end{bmatrix} d\xi = M_o\int_0^1 \begin{bmatrix} 0 \\ \omega_3'(\xi)\delta(\xi-0.5)/L \\ \omega_5'(\xi)\delta(\xi-0.5) \\ 0 \\ \omega_4'(\xi)\delta(\xi-0.5)/L \\ \omega_6'(\xi)\delta(\xi-0.5) \end{bmatrix} d\xi =$$

$$= M_o \begin{bmatrix} 0 \\ -\dfrac{3}{2L} \\ -\dfrac{1}{4} \\ 0 \\ \dfrac{3}{2L} \\ -\dfrac{1}{4} \end{bmatrix}$$

which means that

$$F_{ix} = 0, \qquad F_{iy} = -\frac{3}{2L}M_o, \qquad M_i = -\frac{1}{4}M_o,$$

$$F_{jx} = 0, \qquad F_{jy} = \frac{3}{2L}M_o, \qquad M_j = -\frac{1}{4}M_o.$$

Example No 3.

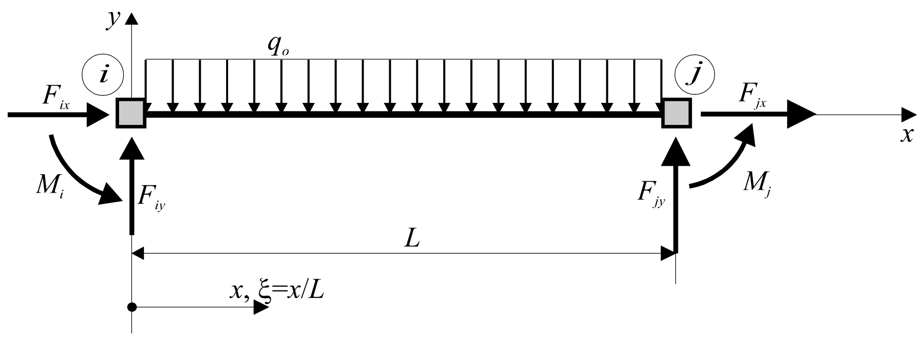

Fig.4.11

The continuous load uniformly distributed on the whole length of an element gives a load vector:

$$\mathbf{q}(\xi) = q_o \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

After inserting the vector $\mathbf{q}(\xi)$ into (4.59), we obtain the equation:

$$\mathbf{f}'^e = q_o L \int_0^1 \begin{bmatrix} 0 \\ \omega_3(\xi) \\ L\omega_5(\xi) \\ 0 \\ \omega_4(\xi) \\ L\omega_6(\xi) \end{bmatrix} d\xi = q_o L \begin{bmatrix} 0 \\ 1/2 \\ L/12 \\ 0 \\ 1/2 \\ -L/12 \end{bmatrix},$$

which means that

$$F_{ix} = 0, \qquad F_{iy} = \frac{1}{2} q_o L, \qquad M_i = \frac{1}{12} q_o L,$$

$$F_{jx} = 0, \qquad F_{jy} = \frac{1}{2} q_o L, \qquad M_j = -\frac{1}{12} q_o L.$$

4.7. FORCES CAUSED BY A TEMPERATURE LOAD

The action of a temperature on frame elements can cause flexion. It happens so when the temperature field is not homogeneous in the cross section. In the case of a truss the flexion of bars did not cause rising nodal forces because truss elements are connected by means of jointed nodes. Bars of frame structures can make a node rotate, hence we have to determine forces at the node in the element undergoing the action of the non-uniform temperature field.

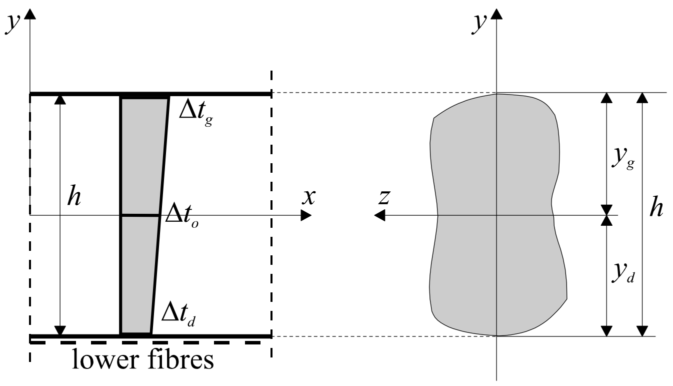

Fig.4.12

Let us consider an element of which the upper fibres are affected by an increase in a temperature Δt_g, and the lower fibres are affected by an increase in a temperature Δt_d (Fig.4.12). The temperature field can be written as follows:

$$\Delta t(x,y) = \Delta t_o(x) + \frac{y}{h}\Delta t_h(x)\,, \tag{4.60}$$

where $\Delta t_o = \frac{1}{h}\left[\Delta t_d y_g + \Delta t_g y_d\right]$ is the increase in the temperature of the middle fibres, $\Delta t_h = \Delta t_g - \Delta t_d$ is the difference of temperatures between extreme fibres, h is the height of the cross section, y_d is the distance between the centre of gravity and the lower fibres, y_g is the distance between the centre of gravity and the upper fibres

Strains of the element fibres induced by the temperature field are equal to

$$\varepsilon_t(y) = \alpha_t \Delta t(y) = \alpha_t\left(\Delta t_o + \Delta t_h \frac{y}{h}\right), \tag{4.61}$$

where α_t is the expansion coefficient of the material.

If bars cannot deform freely, then stresses rise inside them:

$$\sigma_x = -E\varepsilon_t = -\alpha_t E\left(\Delta t_o + \Delta t_h \frac{y}{h}\right), \tag{4.62}$$

which the internal forces result from:

$$N = \int_A \sigma_x dA = -\alpha_t E\left(\Delta t_o \int_A dA + \frac{\Delta t_h}{h}\int_A y dA\right). \tag{4.63}$$

Since the second integral occurring in equation (4.63) is the static moment with regard to the z axis which crosses the centre of gravity, hence this moment has to be equal to zero. Thus, we obtain

$$N_t(x) = -\alpha_t \Delta t_o(x) EA\,, \tag{4.64}$$

like in the case of a truss element.

The second internal force caused by temperature stresses is the bending moment:

$$M_t(x) = \int_A -\sigma_x(x) y dA = \alpha_t E\left[\Delta t_o(x)\int_A y dA + \frac{\Delta t_h}{h}\int_A y^2 dA\right]. \tag{4.65}$$

The first integral in the above equation has to be equal to zero similarly to equation (4.63), and the second one is the moment of inertia of the cross section calculated with regard to the middle axis. Thus, we can write an equation describing the bending moment due to temperature stresses as

$$M_t(x) = \frac{\alpha_t \Delta t_h(x)}{h} E J_z, \tag{4.66}$$

where $J_z = \int_A y^2 dA$ is the moment of inertia of the element section with regard to the z axis crossing the centre of gravity of the section.

We calculate forces at nodes making use of the principle of virtual work just as we did it in Sec.4.6:

$$\left(\mathbf{u}^e\right)^{\mathrm{T}} \mathbf{f}'^{et} = \int_0^L [\boldsymbol{\varepsilon}(x)]^{\mathrm{T}} \mathbf{t}_t dx, \tag{4.67}$$

where

$$\mathbf{t}_t = \begin{bmatrix} N_t(x) \\ 0 \\ M_t(x) \end{bmatrix} \tag{4.68}$$

is the vector of the internal forces induced by a temperature. The zero value of the expression in the second row of the vector comes from the fact that the temperature does not cause shearing forces in the elements, $\boldsymbol{\varepsilon}(x)$ is the vector of displacements gradients:

$$\boldsymbol{\varepsilon}(x) = \begin{bmatrix} \dfrac{du_x}{dx} \\ \dfrac{du_y}{dy} \\ \dfrac{d\varphi}{dx} \end{bmatrix} = \mathbf{B}\,\mathbf{u}^e, \tag{4.69}$$

$\mathbf{B}$ is the matrix of derivatives of shape functions:

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_i & \mathbf{B}_j \end{bmatrix}. \tag{4.70}$$

On the basis of equations (4.52) we calculate

$$\mathbf{B}_i(x)=\begin{bmatrix}\frac{1}{L}\omega_1'(\xi) & 0 & 0\\ 0 & \frac{1}{L}\omega_3'(\xi) & \omega_5'(\xi)\\ 0 & \frac{1}{L^2}\omega_3''(\xi) & \frac{1}{L}\omega_5''(\xi)\end{bmatrix},$$

$$\mathbf{B}_j(x)=\begin{bmatrix}\frac{1}{L}\omega_2'(\xi) & 0 & 0\\ 0 & \frac{1}{L}\omega_4'(\xi) & \omega_6'(\xi)\\ 0 & \frac{1}{L^2}\omega_4''(\xi) & \frac{1}{L}\omega_6''(\xi)\end{bmatrix}, \tag{4.71}$$

where $\omega_i(\xi)$, $\omega_i'(\xi)$, $\omega_i''(\xi)$ (i = 1,2 ... 6)) are nondimensional functions given in Tab.4.1.

On the basis of equation (4.67) we calculate components of the nodal forces vector:

$$\mathbf{f}'^{et}=\int_0^L \mathbf{B}^{\mathrm{T}}\,\mathbf{t}_t dx\,. \tag{4.72}$$

After inserting matrix (4.71) into equation (4.72), we obtain

$$\mathbf{f}'^{et}=\alpha_t E\begin{bmatrix}-A\int_{\xi_1}^{\xi_2}\omega_1'(\xi)\Delta t_o(\xi)d\xi\\ \frac{J_z}{hL}\int_{\xi_1}^{\xi_2}\omega_3''(\xi)\Delta t_h(\xi)d\xi\\ \frac{J_z}{h}\int_{\xi_1}^{\xi_2}\omega_5''(\xi)\Delta t_h(\xi)d\xi\\ -A\int_{\xi_1}^{\xi_2}\omega_2'(\xi)\Delta t_o(\xi)d\xi\\ \frac{J_z}{hL}\int_{\xi_1}^{\xi_2}\omega_4''(\xi)\Delta t_h(\xi)d\xi\\ \frac{J_z}{h}\int_{\xi_1}^{\xi_2}\omega_6''(\xi)\Delta t_h(\xi)d\xi\end{bmatrix}, \tag{4.73}$$

where ξ_1 and ξ_2 are nondimensional coordinates of both the beginning and end of the action interval of the temperature load (Fig.4.13).

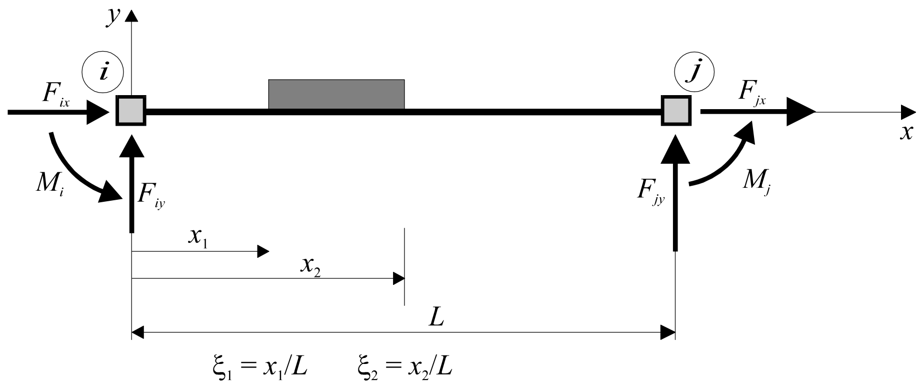

Fig.4.13

In the case when the temperature load is constant and occurs on the whole length of the element, we obtain the following equation from (4.52):

$$\mathbf{f}'^{et} = \alpha_t E \begin{bmatrix} A\Delta t_o \\ 0 \\ -\dfrac{J_z \Delta t_h}{h} \\ -A\Delta t_o \\ 0 \\ \dfrac{J_z \Delta t_h}{h} \end{bmatrix}. \tag{4.74}$$

Both equations (4.52) and (4.53) describe internal forces acting on the element. So when we form the load vector of a structure we should subtract components of this vector from suitable components of the global vector.