CHAPTER II.
STATICS OF 2D TRUSS STRUCTURES

2D trusses are one of the most often used types of structures. The structure of a truss is economic when it respects weight which means that the ratio of the structure weight to forces carried by this structure is expressed in small numbers. It happens so due to building a truss in which, according to assumptions, loads (concentrated forces) will act on nodes only (temperature loads are an exception here) and connection bars will be joined with nodes in an articulated way. Although most constructions which have been built lately are trusses with rigid nodes (they are basically frame constructions which are presented in Chapter IV), methods of solving problems in truss statics with articulated joints are still very important in engineering practice. The system of a plate truss with an articulated joint is the simplest example of an construction showing the idea of the finite element method without employing any complicated details. Hence this chapter will be extended and sometimes too thorough but in subsequent parts of the book we will refer to equations and relations which will be presented here. Moreover, though the structure of the method is very simple, most notions, algorithms and relations connected with the FEM algorithm will be important in discussions of more complex structures.

2.1. BASIC RELATIONS AND NOTATIONS

We assume that the bar of a plate truss (we will also call it an element) is straight and homogeneous (it means that it is made from a homogeneous material without fractures and holes and is with a constant cross section) and it joins nodes i (the first node) and j (the last node). Notations of these nodes (i, j) are local notations which are the same for all bars and they are to define element orientation. On the other hand, structure nodes also have global numbers which allow us to identify them. Global numbers are marked as n_i (the global number of the first node) and n_j (the global number of the last node). The node of a plate truss can move on the plate XY only, in mechanics it means that the node has two degrees of freedom because in order to determine its location during its motion it should be given two coordinates. The situation of the node i of a rigid structure will be determined by initial coordinates X_i, Y_i with respect to the coordinate system which will be used for the description of the whole structure. We say that this system is global and its axes will be noted with capital letters X, Y.

The location of the node i, after its deformation caused by loads, is determined by two components of the displacement vector of nodes u_{iX} and u_{iY}. This method of description of the structure movement is called the Langrange description in mechanics. The description of some dependence between forces and element displacements becomes much simpler when we introduce a local coordinate system which will be noted in small letters x, y. The x axis of the system overlaps the axis of the bar and has its beginning at the first node of an element i, while the y axis is perpendicular to the x axis and is directed in such a way that the Z axis of the global coordinate system and z axis of the local system have the same sense and direction. Because we accept that both coordinate systems are right-torsion, we can obtain the axis y by rotating the x axis clockwise by the angle $\pi/2$.

The most important notations, directions as well as senses of vectors and the coordinate systems are shown in Fig.2.1.

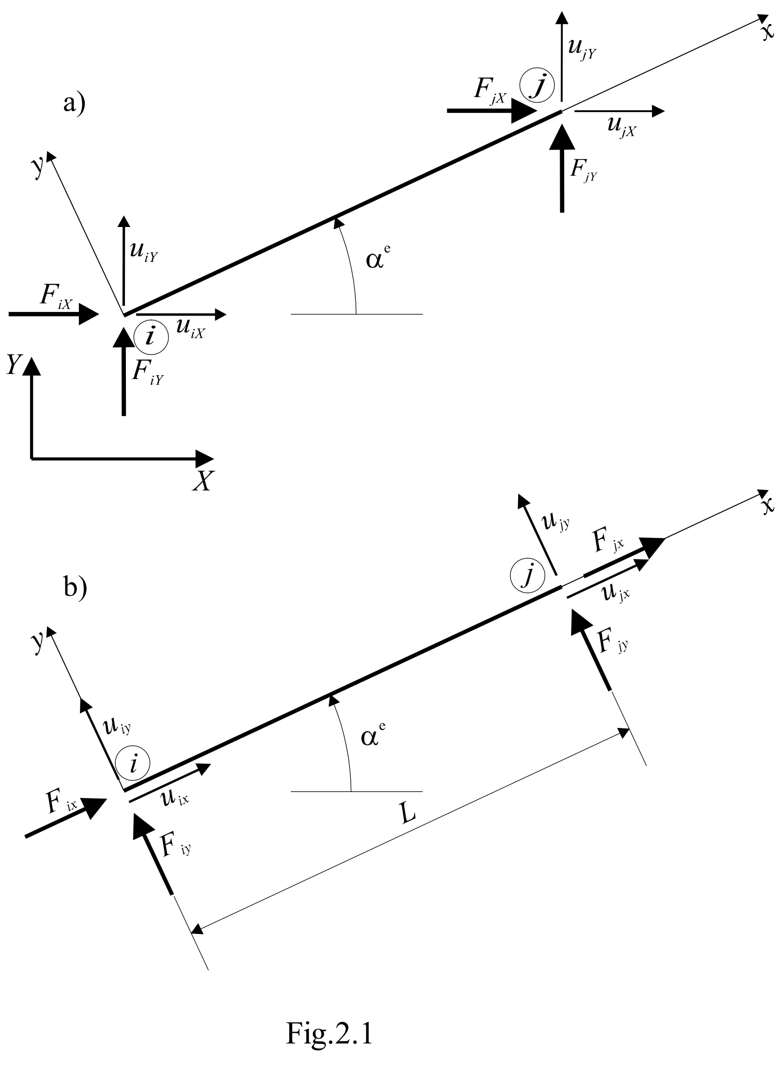


Fig.2.1

Nodal displacements and forces of elements are written as column matrices which we will call vectors. We lead in the following notations:

- The nodal displacement vector of the first node i and the last node j in the local coordinate system:

$$\mathbf{u'}_i = \begin{bmatrix} u_{ix} \\ u_{iy} \end{bmatrix}, \ \mathbf{u'}_j = \begin{bmatrix} u_{jx} \\ u_{jy} \end{bmatrix}. \tag{2.1}$$

- The nodal displacement vector of the element e in the local coordinate system:

$$\mathbf{u'}^e = \begin{bmatrix} \mathbf{u'}_i \\ \mathbf{u'}_j \end{bmatrix} = \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{jx} \\ u_{jy} \end{bmatrix}. \tag{2.2}$$

- The nodal forces vector of the first node i and the last node j in the local coordinate system:

$$\mathbf{f'}_i = \begin{bmatrix} F_{ix} \\ F_{iy} \end{bmatrix}, \ \mathbf{f'}_j = \begin{bmatrix} F_{jx} \\ F_{jy} \end{bmatrix}. \tag{2.3}$$

- The nodal forces vector of the element e in the local coordinate system:

$$\mathbf{f'}^e = \begin{bmatrix} \mathbf{f'}_i \\ \mathbf{f'}_j \end{bmatrix} = \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{jx} \\ F_{jy} \end{bmatrix}. \tag{2.4}$$

2.2. THE ELEMENT STIFFNESS MATRIX OF A PLATE TRUSS IN THE LOCAL COORDINATE SYSTEM

We look for the relation between nodal force vectors and nodal displacement vectors (comp. Chapter I), which is necessary to express equilibrium equations depending on the nodal displacements

$$\mathbf{K'}^e \, \mathbf{u'}^e = \mathbf{f'}^e . \tag{2.5}$$

The general method of building such a relation consists in the use of the principle of virtual work (comp. Chapter I), but in this case we will not apply it and we will use the static analysis which is more clear and possible in the case of such simple elements as bar elements.

Equilibrium equations for the element e (Fig.2.1) lead to the following relations:

$$\begin{aligned} \sum F_x &= F_{ix} + F_{jx} = 0 ; \\ \sum F_y &= F_{iy} + F_{jy} = 0 ; \end{aligned} \tag{2.6}$$

$$\sum M_i = F_{jy}\, L = 0;$$

and we obtain

$$F_{iy} = 0;\; F_{jy} = 0;\; F_{ix} = -F_{jx}. \tag{2.7}$$

Since the set of three equilibrium equations ((2.6) or (2.7)) contains four unknown parameters, this problem is statically indeterminable. The arrangement of an additional equation is necessary in order to make the determination of nodal forces possible. This equation ought to use the relation between nodal displacements of an element and its internal forces. Hooke's law written for a simple case of the axial tension of a straight and homogeneous bar contains these relations (Fig.2.2):

$$\Delta L = \frac{N\,L}{E\,A}, \tag{2.8}$$

where N is the axial force in the bar (the positive value of an axial force always means tension), L is the bar length, ΔL signifies increment of the bar length due to the bar tension caused by the force N; E is Young's modulus of the material from which the bar is made; A is the area of the bar cross section.

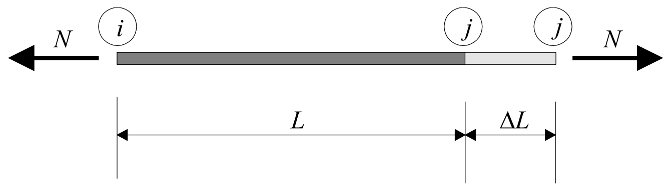


Fig.2.2

Comparing Fig.2.1 and Fig.2.2 we can observe simple relations between nodal forces acting on the bar, that is, F_{ix}, F_{jx} (Fig.2.1) and the axial force N (Fig.2.2):

$$F_{ix} = -N\;;\; F_{jx} = N. \tag{2.9}$$

As it is shown above these relations satisfy the third equilibrium equation (2.7) identically.

The increment of the bar length due to tension results from axial displacements of the bar endings:

$$\Delta L = u_{jx} - u_{ix}, \tag{2.10}$$

which after inserting into equation (2.8) leads to the relation:

$$N = \frac{EA}{L}\left(u_{jx} - u_{ix}\right). \tag{2.11}$$

Taking into consideration the relation between the axial force of the element and nodal forces (2.9) with respect to (2.11) we obtain

$$F_{ix} = \frac{EA}{L}\left(u_{ix} - u_{jx}\right); \; F_{jx} = \frac{EA}{L}\left(-u_{ix} + u_{jx}\right); \tag{2.12a}$$

$$F_{iy} = 0; \; F_{jy} = 0. \tag{2.12b}$$

The obtained relations are the searched relations (2.5) between the nodal forces and nodal displacements of the truss element. We will write them one more time in a different form:

$$\begin{bmatrix} \frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{jx} \\ u_{jy} \end{bmatrix} = \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{jx} \\ F_{jy} \end{bmatrix}. \tag{2.13}$$

After considering notations (2.2), (2.4) and (2.5), the above form leads to the equation:

$$\mathbf{K}'^{e} = \begin{bmatrix} \frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.14}$$

which defines a matrix $\mathbf{K}'^{e}$. This matrix will be called the element stiffness matrix of a plate truss. The matrix in the form of equation (2.14) expresses relations between the vector $\mathbf{u}'^{e}$ and the nodal force vector of an element $\mathbf{f}'^{e}$ in the local coordinate system.

The structure of the stiffness matrix $\mathbf{K}'^{e}$ enables to simplify its transcript:

$$\mathbf{K}'^{e} = \begin{bmatrix} \mathbf{J}' & -\mathbf{J}' \\ -\mathbf{J}' & \mathbf{J}' \end{bmatrix}, \tag{2.15}$$

where $\mathbf{J}'$ is the quadric matrix defined in the following way:

$$\mathbf{J}' = \frac{EA}{L}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.16}$$

2.3. ROTATION OF A NODAL VECTOR -CHANGE OF A COORDINATE SYSTEM

The form of the element stiffness matrix determined in the local coordinate system will not be convenient in further considerations for which we will use matrices of different elements. The most convenient method is importing all matrices to the form which is defined in one common coordinate system. Such a system will be called *the global coordinate system*. It can be the system of a certain type: cartesian, polar or curvilinear. The cartesian coordinate system is the most convenient system for a truss. Nodal coordinates of a structure are usually given in the global coordinate system.

Now we lead the element stiffness matrix to the global system. We start transformations from finding relations for a single node:

$$u_{iX} = u_{ix}\cos\alpha - u_{iy}\sin\alpha\,, \qquad (2.17)$$

$$u_{iY} = u_{ix}\sin\alpha + u_{iy}\cos\alpha\,,$$

or in a matrix form:

$$\begin{bmatrix} u_{iX} \\ u_{iY} \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} u_{ix} \\ u_{iy} \end{bmatrix}, \qquad (2.18)$$

where $c = \cos\alpha$ and $s = \sin\alpha$.

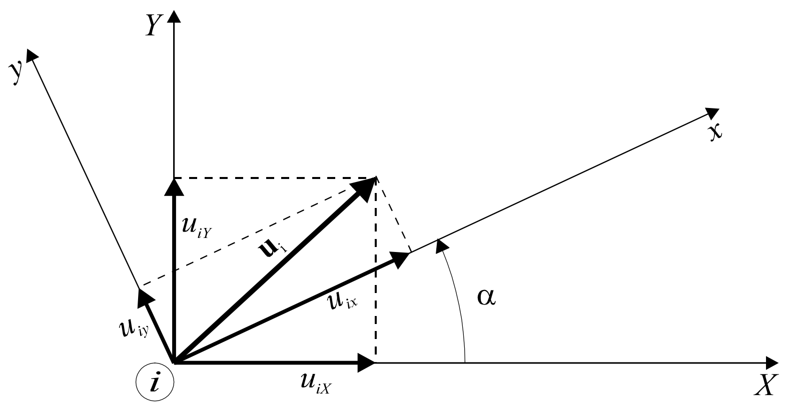


Fig.2.3

Denoting

$$\mathbf{u}_i = \begin{bmatrix} u_{iX} \\ u_{iY} \end{bmatrix} \qquad (2.19)$$

and taking into consideration notation (2.1) we obtain

$$\mathbf{u}_i = \mathbf{R}_i \mathbf{u}'_i\,, \qquad (2.20)$$

where $\mathbf{R}_i = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ (2.21)

is the transformation matrix of the vector $\mathbf{u}'_i$ from the local system to the global one.

A reverse relation will be required:

$$\mathbf{u}'_i = \left(\mathbf{R}_i\right)^{-1} \mathbf{u}_i , \tag{2.22}$$

where $\left(\mathbf{R}_i\right)^{-1}$ is the inverse matrix of $\mathbf{R}_i$; it means that it has such a property that

$$\mathbf{R}_i \left(\mathbf{R}_i\right)^{-1} = \mathbf{I} , \tag{2.23}$$

where **I** is the identity matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \tag{2.24}$$

The matrix $\mathbf{R}_i$ like other transformation matrices has the property that

$$\left(\mathbf{R}_i\right)^{-1} = \left(\mathbf{R}_i\right)^{\mathrm{T}} , \tag{2.25}$$

it means that $\mathbf{R}_i$ is the orthogonality matrix (the determinant of this matrix is equal to 1, it means det($\mathbf{R}_i$)=1; $\det\left(\mathbf{R}_i\right)^{\mathrm{T}} = 1$). We can easily check the property (2.25) of the matrix $\mathbf{R}_i$ making a direct calculation

$$\mathbf{R}_i \left(\mathbf{R}_i\right)^{\mathrm{T}} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \cdot \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} c^2 + s^2 & cs - sc \\ sc - cs & c^2 + s^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I} .$$

The transformation matrix contains the blocks of the nodal transformation matrix:

$$\mathbf{R}^e = \begin{bmatrix} \mathbf{R}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_j \end{bmatrix} , \tag{2.26}$$

where $\mathbf{R}_i$ is the transformation matrix of the first node, $\mathbf{R}_j$ is the transformation matrix of the last node and **0** is the matrix containing zero values. The transformation matrices $\mathbf{R}_i$ and $\mathbf{R}_j$ are usually identical (for straight elements) because rotation angles of the vector of nodes i and j are equal. Since the truss elements are straight, we can write $\mathbf{R}_i = \mathbf{R}_j$.

Finally, the relations between the nodal displacement vector of the element expressed in the local system and the same vector in the global system have the form:

$$\mathbf{u}^e = \mathbf{R}^e \mathbf{u}'^e \tag{2.27}$$

$$\mathbf{u}'^e = \left(\mathbf{R}^e\right)^{\mathrm{T}} \mathbf{u}^e \tag{2.28}$$

The relation between the nodal force vector of an element in the local system and the same vector in the global system is identical to the relation that we have obtained in the equations describing displacements

$$\mathbf{f}_i = \mathbf{R}_i \mathbf{f}'_i \tag{2.29}$$

and

$$\mathbf{f}'_i = \left(\mathbf{R}_i\right)^{\mathrm{T}} \mathbf{f}_i , \tag{2.30}$$

$$\mathbf{f}^e = \mathbf{R}^e \mathbf{f}'^e , \tag{2.31}$$

$$\mathbf{f}'^e = \left(\mathbf{R}^e\right)^{\mathrm{T}} \mathbf{f}^e . \tag{2.32}$$

2.4. THE ELEMENT STIFFNESS MATRIX IN THE GLOBAL COORDINATE SYSTEM

Multiplying equation (2.5) by the transformation of the matrix $\mathbf{R}^e$ and substituting relation (2.28) for $\mathbf{u}'^e$, we obtain

$$\mathbf{R}^e \mathbf{K}'^e \left(\mathbf{R}^e\right)^{\mathrm{T}} \mathbf{u}^e = \mathbf{R}^e \mathbf{f}'^e \tag{2.33}$$

On the basis of relation (2.31) the right hand side of this equation is equal to $\mathbf{f}^e$, so if we lead the notation

$$\mathbf{K}^e = \mathbf{R}^e \mathbf{K}'^e \left(\mathbf{R}^e\right)^{\mathrm{T}} \tag{2.34}$$

we obtain

$$\mathbf{f}^e = \mathbf{K}^e \mathbf{u}^e , \tag{2.35}$$

It is the searched relation between nodal forces and displacements of the element in the global coordinate system.

If we perform multiplication existing in equation (2.34) we obtain

$$\mathbf{K}^e = \begin{bmatrix} \mathbf{J} & -\mathbf{J} \\ -\mathbf{J} & \mathbf{J} \end{bmatrix} , \tag{2.36}$$

where $\mathbf{J} = \dfrac{EA}{L} \begin{bmatrix} c^2 & sc \\ sc & s^2 \end{bmatrix}$. (2.37)

We can exchange form (2.37) of the matrix $\mathbf{J}$ into the equivalent one in which trigonometric functions do not exist. Let us note that

$$c = \cos\alpha = \frac{L_X}{L} \quad \text{and} \quad s = \sin\alpha = \frac{L_Y}{L} . \tag{2.38}$$

After inserting these relations into (2.37), we obtain

$$\mathbf{J} = \frac{EA}{L^3}\begin{bmatrix} L_X^2 & L_X L_Y \\ L_X L_Y & L_Y^2 \end{bmatrix} \tag{2.39}$$

Example 2.E1 shows the use of equation (2.37) for determining coefficients of element stiffness matrices for a plate truss.

2.5. NODAL EQUILIBRIUM EQUATIONS AND AGGREGATION OF A STIFFNESS MATRIX

Replacing existing bars (elements) of a truss by nodal forces we obtain a group of nodes which can be treated as material particles with two degrees of freedom. These nodes are loaded with concentrated forces coming from elements or external loads. The equilibrium conditions for such a node we write as follows:

$$\sum P_X = \sum_{k=1}^{E_n}\left(-F_{nX}^{e_k}\right) + P_{nX} = 0\,,$$
$$\sum P_Y = \sum_{k=1}^{E_n}\left(-F_{nY}^{e_k}\right) + P_{nY} = 0\,, \tag{2.40}$$

where we have noted $F_{nX}^{e_k}$ - component in the direction X of nodal forces from the element numbered e_k acting on a node n, P_{nX} - component in the direction X of the external forces acting on the node n, E_n - number of elements joined to the node n.

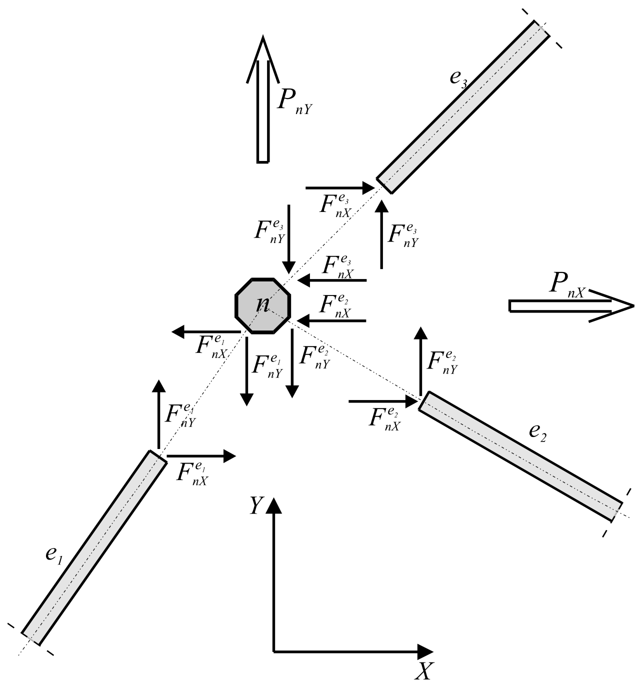


Fig.2.4

Now we are transforming the set of equations (2.40) to the form containing nodal displacements:

$$\begin{bmatrix} \mathbf{K}_{1n} & \mathbf{K}_{2n} & \dots & \mathbf{K}_{in} & \dots & \mathbf{K}_{N_n n} \end{bmatrix} \mathbf{u} = \mathbf{p}_n \tag{2.41}$$

In equation (2.41)

$\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_i \\ \vdots \\ \mathbf{u}_{N_n} \end{bmatrix}$ signifies the global vector of nodal displacements of a structure,

$\mathbf{p}_n = \begin{bmatrix} P_{nX} \\ P_{nY} \end{bmatrix}$ is the vector of external forces acting on the node n,

matrices $\mathbf{K}_{in}$ are quadratic matrices with dimensions 2x2 determined as follows:

where $i{=}n$ - $\mathbf{K}_{nn} = \sum_{k=1}^{E_n} \mathbf{J}^{e_k}$, (2.42)

$e_1, e_2 \dots e_k \dots e_{E_n}$ - are numbers of elements joined to the node n,

if $i \neq n$ and nodes i and n are not directly connected by any elements, then $\mathbf{K}_{in} = 0$,

if $i \neq n$ and nodes i and n are connected by some element with a number e, then $\mathbf{K}_{in} = -\mathbf{J}^e$,

$\mathbf{J}^e$ - signifies the block of a stiffness matrix of the element e (comp. equation (2.37)).

Arranging equilibrium equations (2.41) for all nodes of a structure we obtain the final form of equations serving determination of nodal displacements of the truss:

$$\left[\begin{array}{c:c:c:c:c:c} \mathbf{K}_{11} & \mathbf{K}_{12} & \cdots & \mathbf{K}_{1n} & \cdots & \mathbf{K}_{1N_n} \\ \hdashline \mathbf{K}_{21} & \mathbf{K}_{22} & \cdots & \mathbf{K}_{2n} & \cdots & \mathbf{K}_{2N_n} \\ \hdashline \vdots & \vdots & \ddots & \vdots & & \vdots \\ \hdashline \mathbf{K}_{n1} & \mathbf{K}_{n2} & \cdots & \mathbf{K}_{nn} & \cdots & \mathbf{K}_{nN_n} \\ \hdashline \vdots & \vdots & & \vdots & \ddots & \vdots \\ \hdashline \mathbf{K}_{N_n 1} & \mathbf{K}_{N_n 2} & \cdots & \mathbf{K}_{N_n n} & \cdots & \mathbf{K}_{N_n N_n} \end{array}\right] \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_n \\ \vdots \\ \mathbf{u}_{N_n} \end{bmatrix} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \vdots \\ \mathbf{p}_n \\ \vdots \\ \mathbf{p}_{N_n} \end{bmatrix}$$

or $\mathbf{K}\,\mathbf{u} = \mathbf{p}$ (2.43)

The matrix **K** of the set of equations (2.43) is the global stiffness matrix of the structure, the vector **u** is the global vector of nodal displacements of the structure, the vector **p** is the global vector of nodal forces of the structure.

Proper numbering nodes can lead the matrix **K** to the banded matrix (comp. Example 2.E1) which is characterised by a fact that non-zero components appear on the main diagonal and closely to it. The matrix **K** is a symmetric matrix which means that its components satisfy equations:

$$K_{ij} = K_{ji} \text{ or } \mathbf{K} = \mathbf{K}^{\mathrm{T}} \tag{2.44}$$

which result from the principle of virtual work (comp. Chapter I). Components K_{nn} which are on the main diagonal are always positive

$$K_{nn} > 0 \tag{2.45}$$

which is a direct conclusion drawn from definitions (2.37) and (2.42).

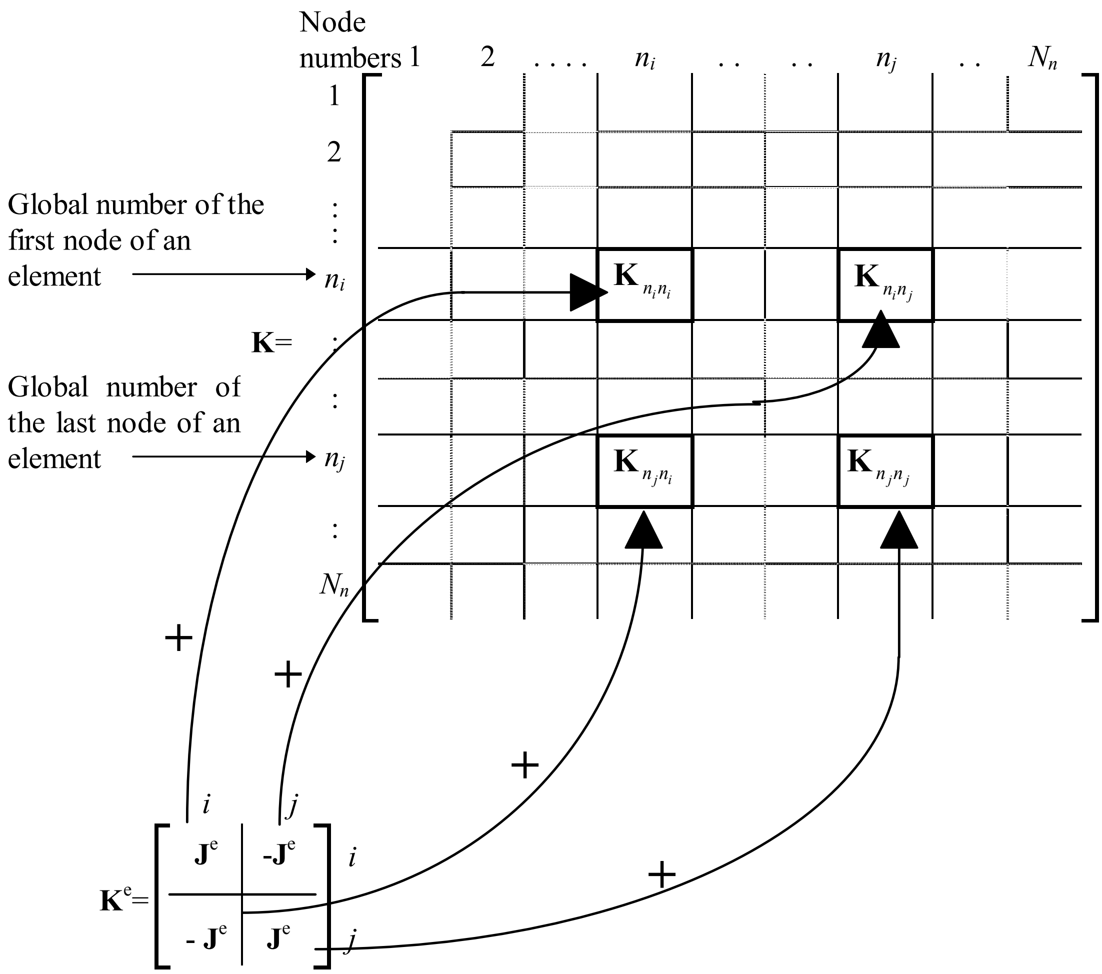


Fig.2.5

The zero component K_{nn} testifies geometric changability of a structure and should be removed by a suitable change of a geometric scheme. The matrix **K** presented by equation (2.43) is a singular matrix (it means det **K**=0), hence the set of equations (2.43) cannot be solved without modifying it. This modification will depend on the consideration of boundary conditions. We will occupy with this problem in the next section.

The process of building the global stiffness matrix is called aggregation of a matrix. It can be done by means of the method described in Chapter I demanding formation of connection matrices. Since these matrices are large, then their use is not convenient and they are rarely used in computer implementation of the FEM algorithm. The method of summation of blocks shown by equations (2.41) and (2.42) is much simpler. The form of matrix equations (2.41) and (2.42) may seem to be complicated, but in fact, we have very simple operations of insertion of blocks here. This method is best shown in Fig.2.5.

Signs "+" located at arrows pointing to the place of location of blocks $\mathbf{K}^e$ mean that blocks $\mathbf{J}^e$ should be added to the existing contents of „cells" of matrices $\mathbf{K}_{n_i n_i}$ or $\mathbf{K}_{n_j n_j}$, and blocks $-\mathbf{J}^e$ lying beyond the diagonal should be added to „cells" $\mathbf{K}_{n_i n_j}$ or $\mathbf{K}_{n_j n_i}$. In the case of a truss where nodes are usually joined by one element, blocks lying beyond the main diagonal contain only a single matrix $-\mathbf{J}^e$. But blocks lying on the main diagonal $\mathbf{K}_{n_i n_i}$ contain sums of as many matrices $\mathbf{J}^e$ as many elements are joined with the node n_i. An adequate example explaining the technique of aggregation of the stiffness matrix is contained in example 2.E1.

2.6. BOUNDARY CONDITIONS

As it was noted in the previous section of this Chapter the global stiffness matrix of a structure is most often a singular matrix directly after the aggregation. It means that the determination of this matrix is equal zero. Because the set of equations (2.43) has to have only one solution for static problems, we have to modify the global stiffness matrix. It should be done in such a way that the solution of the set of linear equations (2.43) is possible. The reason for singularity of the matrix **K** is the lack of information about supports of the construction. We have never used information about support conditions, thus we ought to define what the support of the node is.

For trusses there are two types of supports possible: an articulated support and an articulated movable support. The articulated support (scheme of this support is shown in Fig.2.6a) prevents movements of a node in any direction which means:

$$u_{rX} = 0,\ u_{rY} = 0. \tag{2.46}$$

The movement of the support node r causes reactions in two components: R_X and R_Y (Fig.2.6a), which counteract the movement of the node r. We say that this support assures *free support* of a node.

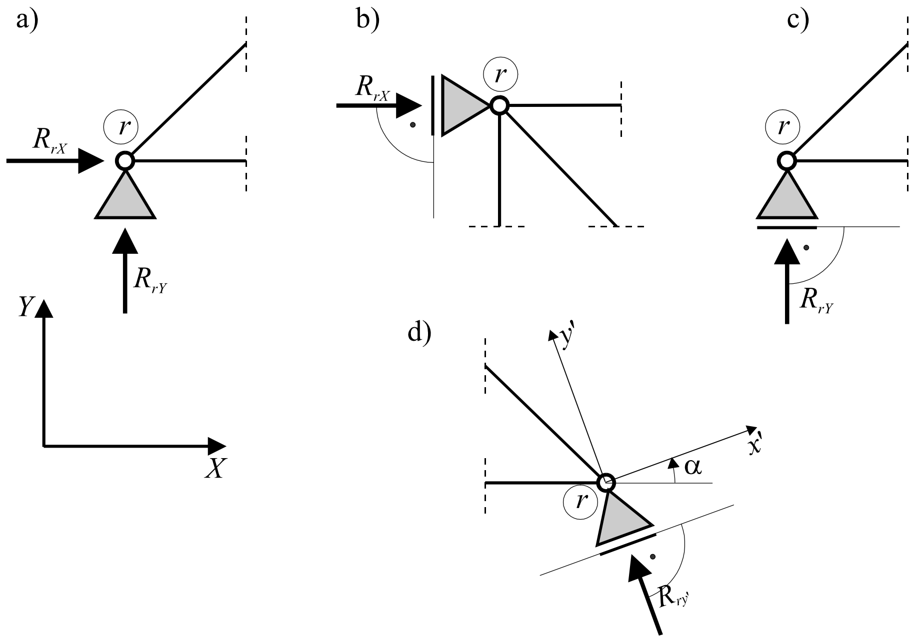


Fig.2.6. Types of supports of a plate truss

The next support shown in Fig.2.6b is called an articulated movable support and it prevents movements of a node along one line only, then it enables the movement of a node in perpendicular direction with respect to this line. The reaction occurring in the articulated movable support can have the direction of this line only (Fig.2.6 b, c, d). The support can appear in a few variants, two most often occurring variants (shown in Fig.2.6 b, c) give very simple support conditions:

- support with the possibility of a movement along the Y axis of the global coordinate system (Fig.2.6 b)

$$\mathbf{u}_{rX} = 0, \tag{2.47}$$

- support with the possibility of a movement along the X axis of the global coordinate system (Fig.2.6 c)

$$\mathbf{u}_{rY} = 0. \tag{2.48}$$

The third variant of a movable support causes us some problem with describing the boundary conditions because the direction of the reaction of this support (Fig.2.6 d) is not parallel to any axis of the global coordinate system. It is important because equilibrium equations (2.40) leading to equation (2.43) were written in the global coordinate system. In the case of a support with a movement not parallel to any axis of the global coordinate system (we will call such supports skew supports) we have to renounce this convenient manner and we have to write the boundary conditions in the system *x'y'* connected with the support. The system *x'y'* is rotated with respect to the global system by an angle α' (Fig.2.6d). We will elaborate the transformation method for a set of equations at a support node to the local system in the next section. Now we will focus on describing the boundary condition. We write the condition of absence of a movement along the *y'* axis analogously as in equation (2.48):

$$\mathbf{u}_{ry'} = 0. \tag{2.49}$$

Equations ((2.46) ... (2.49)) describing the boundary conditions give us the values of displacements in support nodes. Hence some equations of set (2.43) should be removed, because they contain unknown forces acting on support nodes (constraint reactions). These equations can be replaced by equations of boundary conditions (for example (2.46)). It is usually done by modifying some equations (2.43).

Let *m* mean the global number of a degree of freedom which is eliminated by the boundary condition: $u_m = 0$, then we modify the row with the number *m* in the global stiffness matrix **K**, replacing it by a row containing zeros and the value 1 in the column *m*:

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} & \cdots & \mathbf{K}_{1m} & \cdots & \mathbf{K}_{1N_n} \\ \mathbf{K}_{21} & \mathbf{K}_{22} & \cdots & \mathbf{K}_{2m} & \cdots & \mathbf{K}_{2N_n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ \mathbf{K}_{N_n1} & \mathbf{K}_{N_n2} & \cdots & \mathbf{K}_{N_nm} & \cdots & \mathbf{K}_{N_nN_n} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \\ \vdots \\ u_{N_n} \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ 0 \\ \vdots \\ P_{N_n} \end{bmatrix}$$

or $\mathbf{K}^o\,\mathbf{u} = \mathbf{p}^r$. (2.50)

The nodal load vector **p** should be modified so that equation *m* contains zero on the right side. The modified matrices are marked in equation (2.50) by an upper index *r*.

These changes in the stiffness matrix disturb the symmetry of it because $K_{im} \neq 0$ but $K_{mi} = 0$ when $i \neq m$ (comp. (2.50)). The absence of symmetry in the stiffness matrix does not prevent solving of equilibrium equations (2.43) but it considerably loads the computer memory storing coefficients K_{ij} either in the core memory (RAM) or external space (disk) which lengthens the solution time for a set of equations (comp. Appendix 2). Thus, let us try to restore the symmetry of the matrix $\mathbf{K}^o$ (2.50). Let us note that the terms located in the column with the number m are multiplied by the zero value of the displacement u_m. Hence we can insert zeros instead of coefficients in the column m (except for one coefficient in the row m which has to be equal to 1). If we modify the stiffness matrix in that way, the solution of our problem will be the same and the matrix will be a symmetric one:

$$\mathbf{K}^r = \left[\begin{array}{c:c:c:c:c:c} \mathbf{K}_{11} & \mathbf{K}_{12} & \cdots & 0 & \cdots & \mathbf{K}_{1N_n} \\ \hdashline \mathbf{K}_{21} & \mathbf{K}_{22} & \cdots & 0 & \cdots & \mathbf{K}_{2N_n} \\ \hdashline \vdots & \vdots & \ddots & \vdots & & \vdots \\ \hdashline 0 & 0 & \cdots & 1 & \cdots & 0 \\ \hdashline \vdots & \vdots & & \vdots & \ddots & \vdots \\ \hdashline \mathbf{K}_{N_n 1} & \mathbf{K}_{N_n 2} & \cdots & 0 & \cdots & \mathbf{K}_{N_n N_n} \end{array}\right] \tag{2.51}$$

Finally, we solve the problem:

$$\mathbf{K}^r \mathbf{u} = \mathbf{p}^r, \tag{2.52}$$

where the matrix $\mathbf{K}^r$ is symmetrical and is not singular which means that $\det \mathbf{K}^r \neq 0$, if we have properly chosen the boundary conditions. On the basis of the theorem about the positive value of a strain energy (comp. equation (1.45), Chapter I) we can conclude that the matrix $\mathbf{K}^r$ has to be positive-determined, then

$$\det \mathbf{K}^r > 0. \tag{2.53}$$

Hence the set of equations (2.52) has one solution.

In small finite element systems (programmes) the matrix $\mathbf{K}^r$ is usually left in the form noted in equation (2.51). Large and complex systems prepared to solve problems described by many thousands of equations usually remove rows and columns containing zeros from the matrix $\mathbf{K}^r$ and vector $\mathbf{p}^r$. It is done to reduce dimensions of a solved problem. This method of modification of the matrix $\mathbf{K}^r$ needs new numbering of degrees of freedom of a structure. Because it is not strictly joined with FEM and is connected with the computer implementation of the FEM algorithm, we will not describe it here.

2.7. TRANSFORMATION OF THE STIFFNESS MATRIX FOR A „SKEW" SUPPORT

Now we are elaborating ways of transformation of an element stiffness matrix joined to a support node by means of a „skew" support (Fig.2.6d). We chose the coordinate system $x'y'$ in such a way that the direction of a support reaction covers the y' axis and a movement line will be parallel to the x' axis (an opposite choice of the local coordinate system is obviously possible). The x' axis is rotated with respect to the X axis of the global system by the angle α' which we will deem to be positive when the rotation from the X axis to the x' axis will be anticlockwise. The positive angle α' is shown in Fig.2.6 d.

If we write equilibrium equations for the support node r in the system $x'y'$, then the boundary condition of this support is determined by equation (2.49). Let us try to perform the necessary transformation. We make use of relations (2.20) and (2.22) which served us in Sec.2.3 to pass from the local system of an element to the global one.

Then we express the nodal forces vector at the node r as follows:

$$\begin{bmatrix} F_{rx'} \\ F_{ry'} \end{bmatrix} = \begin{bmatrix} c' & s' \\ -s' & c' \end{bmatrix} \begin{bmatrix} F_{rX} \\ F_{rY} \end{bmatrix},$$

or in a shorter form:

$$\mathbf{f'}_r = \left(\mathbf{R'}_r\right)^{\mathrm{T}} \mathbf{f}_r . \tag{2.54}$$

Next we transform the nodal displacements vector of the support node from the local system to the global one as follows:

$$\begin{bmatrix} u_{rX} \\ u_{rY} \end{bmatrix} = \begin{bmatrix} c' & -s' \\ s' & c' \end{bmatrix} \begin{bmatrix} u_{rx'} \\ u_{ry'} \end{bmatrix},$$

or in a close form:

$$\mathbf{u}_r = \mathbf{R'}_r \, \mathbf{u'}_r . \tag{2.55}$$

In equations (2.54) and (2.55)we have marked

$$\mathbf{R'}_r = \begin{bmatrix} c' & -s' \\ s' & c' \end{bmatrix}, \; c' = \cos\alpha', \; s' = \sin\alpha'$$

and $\left(\mathbf{R'}_r\right)^{\mathrm{T}}$ is the transpose of the matrix $\mathbf{R'}_r$.

Let us assume that an element e joins nodes r_i and r_j supported by „skew" supports which are rotated by angles α'_i and α'_j (Fig.2.6). Then we write equilibrium equations for nodes r_i and r_j in the local coordinate system $x'_i\, y'_i$ at the node r_i and $x'_j\, y'_j$ at the node r_j. The transformation of nodal forces vectors and nodal displacements vectors of the element e looks as follows:

- for a nodal forces vector

$$\mathbf{f}'^e = \left(\mathbf{R}'^e\right)^{\mathrm{T}} \mathbf{f}^e \tag{2.56}$$

or in a developed form

$$\begin{bmatrix} \mathbf{f}'_{r_i} \\ \mathbf{f}'_{r_j} \end{bmatrix} = \begin{bmatrix} \left(\mathbf{R}'_{r_i}\right)^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \left(\mathbf{R}'_{r_j}\right)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{f}_{r_i} \\ \mathbf{f}_{r_j} \end{bmatrix},$$

- for the nodal displacements vector

$$\mathbf{u}^e = \mathbf{R}'^e\, \mathbf{u}'^e \tag{2.57}$$

or $\begin{bmatrix} \mathbf{u}_{r_i} \\ \mathbf{u}_{r_j} \end{bmatrix} = \begin{bmatrix} \mathbf{R}'_{r_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}'_{r_j} \end{bmatrix} \begin{bmatrix} \mathbf{u}'_{r_i} \\ \mathbf{u}'_{r_j} \end{bmatrix}.$

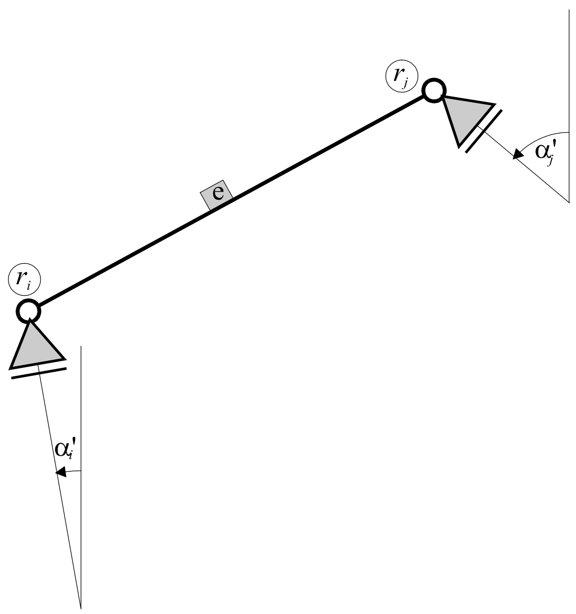


Fig.2.7

Inserting relation (2.57) into (2.35) and next the obtained results into (2.56), we get the equation transforming the stiffness matrix of the element e from the global coordinate system to the support coordinate system:

$$\mathbf{f}'^e = \left(\mathbf{R}'^e\right)^{\mathrm{T}} \mathbf{K}^e \mathbf{R}'^e\, \mathbf{u}'^e \tag{2.58}$$

We simplify this equation to the form:

$$\mathbf{f}'^{e} = \mathbf{K}'^{e}\,\mathbf{u}'^{e}, \tag{2.59}$$

in which we make use of the substitution:

$$\mathbf{K}'^{e} = \left(\mathbf{R}'^{e}\right)^{\mathrm{T}}\mathbf{K}^{e}\mathbf{R}'^{e}, \tag{2.60}$$

defining the element matrix in the support coordinate system.

One of angles α' (Fig.2.7) is most often equal to zero because it rarely happens that a truss bar joins two support nodes supported by a „skew" support. The transformation matrix of a zero angle is a unit matrix. Because (c'=1, s'=0), then the element transformation matrix is simplified to the form:

$$\mathbf{R}'^{e} = \begin{bmatrix} \mathbf{R}'_{r_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \tag{2.61}$$

when the second node is described in the global system but we transform forces and displacements at the first node r_i, and

$$\mathbf{R}'^{e} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}'_{r_j} \end{bmatrix}, \tag{2.62}$$

when the transformation concerns the last node r_j only.

As it has been shown, the existence of „skew" supports complicates the simple FEM algorithm presented in Chapter I because it requires additional transformations of element stiffness matrices before the aggregation of the global matrix is done. There are some other simpler though approximate methods of solving this problem and they will be discussed in the next section concerning boundary elements.

2.8. ELASTIC SUPPORTS AND BOUNDARY ELEMENTS

Not all kinds of supports applied to support trusses can be described by the boundary conditions of types (2.47), (2.48) and (2.49). There are flexible supports which have displacements connected with a support reaction, for instance, the linear relation of the following type:

$$\begin{aligned} R_{rX} &= -h_{rX}u_{rX}, \\ R_{rY} &= -h_{rY}u_{rY}, \end{aligned} \tag{2.63}$$

where h_{rX} is the support stiffness in the direction of the X axis and h_{rY} is the support stiffness in the direction of the Y axis. The linear spring shown in Fig.2.8 is a good model of this type of support.

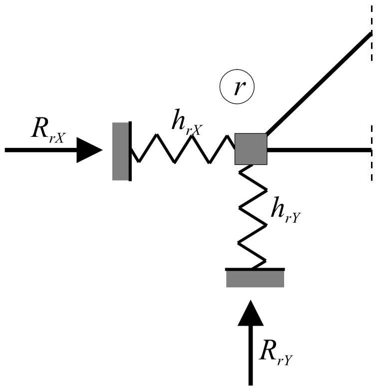


Fig.2.8

If we treat reactions R_{rX} and R_{rY} acting on the node supported elastically as external forces, then we obtain the nodal forces vector containing unknown displacements u_{rX}, u_{rY}:

$$
\mathbf{p}=\begin{array}{c}\left[\begin{array}{c} P_{1X} \\ P_{1Y} \\ \hline P_{2X} \\ P_{2Y} \\ \hline \vdots \\ \hline R_{rX} \\ R_{rY} \\ \hline \vdots \\ \hline P_{N_nX} \\ P_{N_nY} \end{array}\right]\end{array}\begin{array}{c} 1 \\ \\ 2 \\ \\ \vdots \\ r \\ \\ \vdots \\ N_n \\ \\ \end{array} = \left[\begin{array}{c} P_{1X} \\ P_{1Y} \\ P_{2X} \\ P_{2Y} \\ \vdots \\ -h_{rX}u_{rX} \\ -h_{rY}u_{rY} \\ \vdots \\ P_{N_nX} \\ P_{N_nY} \end{array}\right] \tag{2.64}
$$

The vector $\mathbf{p}$ cannot be absolutely used as the right hand side of equation (2.43) where unknown values of nodal displacements should be on the left hand side of the equation. Now we are transforming the vector $\mathbf{p}$ described by equation (2.64) in such a way that nodal reactions of the elastic node r will be moved to the left hand side of the equilibrium equation:

$$\mathbf{K}^s\mathbf{u} = \mathbf{p}^r\,, \tag{2.65}$$

where $\mathbf{K}^s$ is the stiffness matrix containing information about elastic supports of the structure and $\mathbf{p}^r$ is the nodal forces vector in which the boundary conditions written in equation (2.50) (we can treat the elastic supports as fixed ones after transferring describing them relations to the left hand side of the equation) are considerated.

The matrix $\mathbf{K}^s$ is written by the equation:

$$\mathbf{K}^s = \left[\begin{array}{cc:c:cc:c:c}
\mathbf{K}_{11} & \mathbf{K}_{12} & & \mathbf{K}_{1m} & \mathbf{K}_{1(m+1)} & & \mathbf{K}_{1N_n} \\
\mathbf{K}_{21} & \mathbf{K}_{22} & \cdots & \mathbf{K}_{2m} & \mathbf{K}_{2(m+1)} & \cdots & \mathbf{K}_{2N_n} \\
\hdashline
\vdots & & \ddots & & \vdots & & \vdots \\
\hdashline
\mathbf{K}_{m1} & \mathbf{K}_{m2} & & \mathbf{K}_{mm} + h_{rX} & \mathbf{K}_{m(m+1)} & & \mathbf{K}_{mN_n} \\
\mathbf{K}_{(m+1)1} & \mathbf{K}_{(m+1)2} & \cdots & \mathbf{K}_{(m+1)m} & \mathbf{K}_{(m+1)(m+1)} + h_{rY} & \cdots & \mathbf{K}_{(m+1)N_n} \\
\hdashline
\vdots & & & & \vdots & \ddots & \vdots \\
\hdashline
\mathbf{K}_{N_n1} & \mathbf{K}_{N_n2} & \cdots & \mathbf{K}_{N_nm} & \mathbf{K}_{N_n(m+1)} & \cdots & \mathbf{K}_{N_nN_n}
\end{array}\right]
\begin{array}{c}
1 \\ \\ \vdots \\ r \\ \\ N_n \\ \\
\end{array}
\tag{2.66}$$

where m is the global number of the first degree of freedom of the node r. With standard numbering $m=(r-1)N_D+1$ N_D is the number of degrees of freedom of the node. For a 2D truss $N_D=2$, the number of the first degree of freedom of the node r is equal to $m=2r-1$.

At this stage the modified matrix $\mathbf{K}^s$ contains the stiffness of elastic supports which are added to the terms coming from the truss element of a construction. These sums are located on the main diagonal of the matrix in rows describing the equilibrium of the node r. Such an interpretation of elastic supports leads to a convenient although simplistic way of considering fixed supports. We substitute them for elastic supports with very large stiffness inserting the numbers equal to for example $H=1\cdot 10^{30}$ onto the main diagonal. This method was formulated by Irons [7] who multiplies terms lying in a suitable row on the diagonal of the matrix $\mathbf{K}$ by numbers of the order of 10^6. After inserting a high value onto the diagonal, it is redundant to insert zeros both in rows and columns of the matrix $\mathbf{K}$ as well as rows of the vector of the right hand side $\mathbf{p}$. It is very important for large stiffness matrices which are often stored in structures of data different from quadratic tables (comp. Appendix 2). The simplicity of this way causes that it is commonly used in the computer implementation of the FEM algorithm instead of the exact method described in Sec.2.6.

Elastic supports also suggest the use of a special support element which could substitute any elastic constraints and fixed supports (which should be treated as elastic supports with large stiffness). This support element rotated by an angle α with respect to the global coordinate system is shown in Fig.2.9.

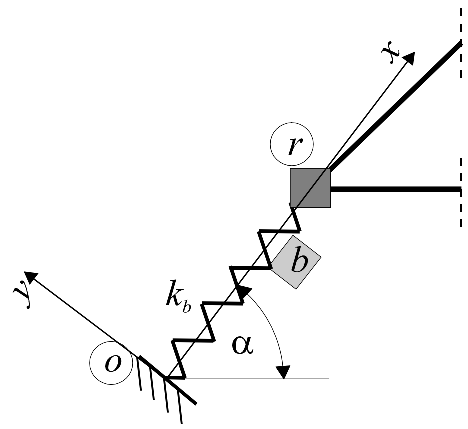


Fig.2.9

We can easily obtain the stiffness matrix of such an element from the matrix of an ordinary truss element described by equation (2.14) in the local coordinate system or equation (2.36) in the global system. We do it in such a way that we substitute the stiffness of a bar EA/L for the stiffness of the elastic boundary element k_b. In general, the node o of this element is always fixed, so we can remove it from the set of equations which allows us to treat the boundary element as an element with two degrees of freedom:

$$\mathbf{K}^b = k_b \begin{bmatrix} c^2 & sc \\ sc & s^2 \end{bmatrix}, \tag{2.67}$$

where similarly to equation (2.37) $c = \cos\alpha$, $s = \sin\alpha$.

When we want to substitute the fixed support for this element we accept $k_b=H$. The value of the number H depends on the computer system in which the programme will be started and most of all it depends on the type of real numbers. We can take for example $H=1\cdot 10^{30}$ as reference for many systems.

2.9. THE NODAL LOADS VECTOR WITH TEMPERATURE LOAD

As we have already noted in the introduction to this Chapter, unique loads of a truss which act on elements and do not act on nodes directly are temperature loads. Now we are showing how we can lead this load to known to us loads, that is, concentrated forces acting on the nodes of a construction.

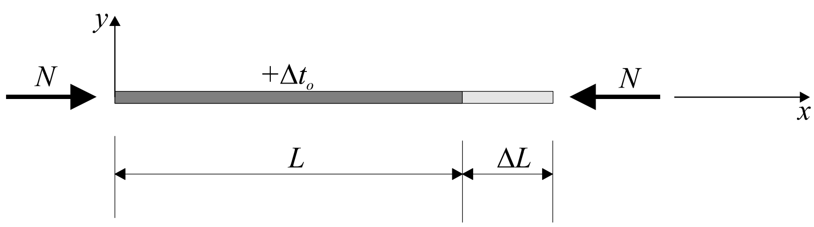


Fig.2.10

As we know the increase in a temperature of an element causes its lengthening which with the assumption of a steady increase in a temperature on the whole length of a bar is described by the equation:

$$\varepsilon_t = \frac{\Delta L}{L} = \alpha_t \Delta t_o, \tag{2.68}$$

where α_t is the coefficient of thermal expansion of the material from which the element is made, Δt_o stands for an increment of a temperature in middle fibres (joining centres of gravity of cross sections of an element).

We assume a steady increase in a temperature in the whole section and homogeneity of the material. If we accept that the element has no freedom of lengthening but is limited by fixed nodes, we obtain an axial force which rises within the element:

$$N = -\int_A \sigma_t d = -\int_A E\varepsilon_t dA = -\int_A E\alpha_t \Delta t_o dA = -E\alpha_t \Delta t_o A, \tag{2.69}$$

where E is Young's modulus of the material and A signifies the surface area of the cross section of the element.

The nodal forces vector of the element due to the temperature written in the local coordinate system xy is equal to:

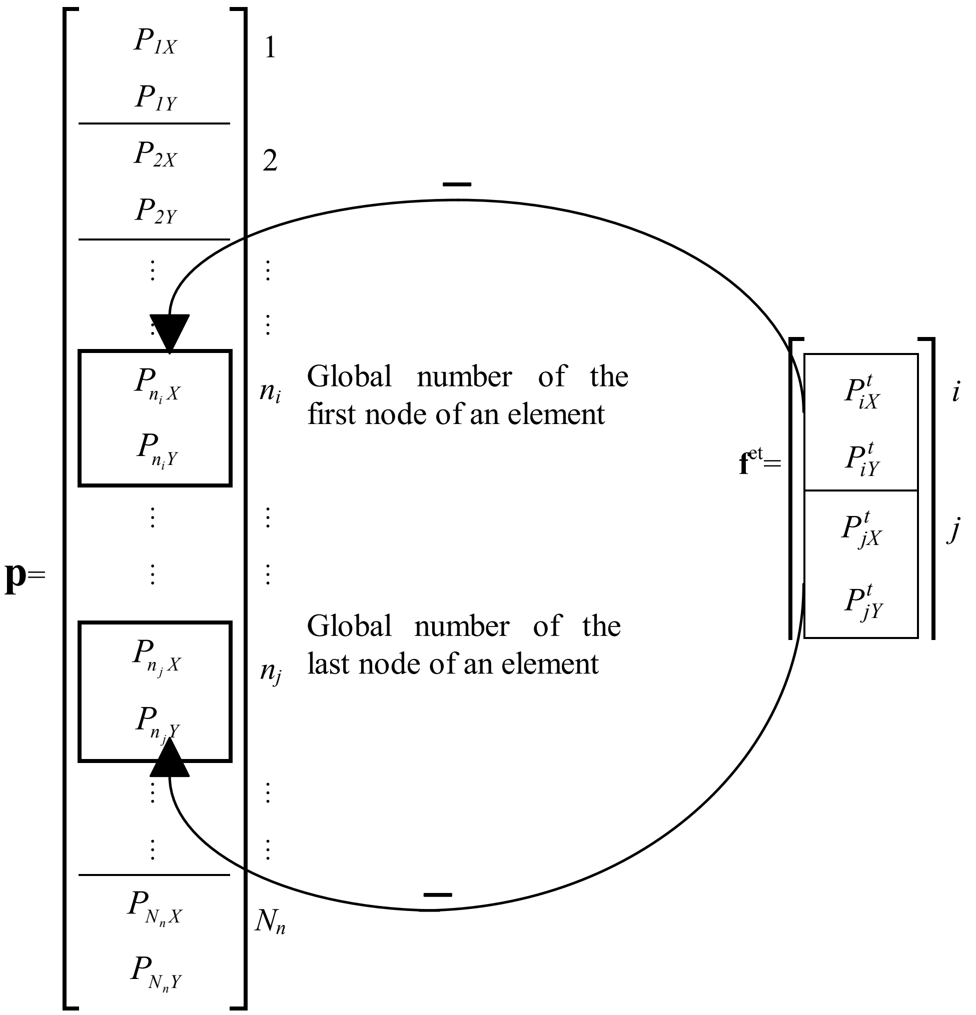
$$\mathbf{f}'^{et} = EA\alpha_t \Delta t_o \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \tag{2.70}$$

after transformation to the global system with the help of relation (2.31) we obtain

$$\mathbf{f}^{et} = EA\alpha_t \Delta t_o \begin{bmatrix} c \\ s \\ -c \\ -s \end{bmatrix}, \tag{2.71}$$

where $c = \cos\alpha$, $s = \sin\alpha$, α - is the angle determining a slope of the loaded element with respect to the global coordinate system.

Since forces acting on the nodes are necessary to arrange equilibrium equations, and as it is known, they are reversely directed to other forces acting on elements, then while building the global nodal forces vector we subtract them from other forces. It is shown in Fig.2.11.



$$P_{iX}^t = EA\alpha_t \Delta t_o \cos\alpha \qquad P_{jX}^t = -EA\alpha_t \Delta t_o \cos\alpha$$

$$P_{iY}^t = EA\alpha_t \Delta t_o \sin\alpha \qquad P_{jY}^t = -EA\alpha_t \Delta t_o \sin\alpha$$

Fig.2.11

2.10. THE GEOMETRIC LOAD OF A TRUSS

The final type of a truss load which we will describe is the geometric load (forced displacements of nodes).

We assume that the node r is displaced by the vector **d** (Fig.2.12). Surely, it is necessary to apply forces to the node to cause this displacement. Values of these forces are not known, whereas we know components of the displacement of the node r:

$$u_{rX} = d_X\,,\ u_{rY} = d_Y\,, \tag{2.72}$$

where d_X, d_Y are the components of the vector of the forced displacement **d**.

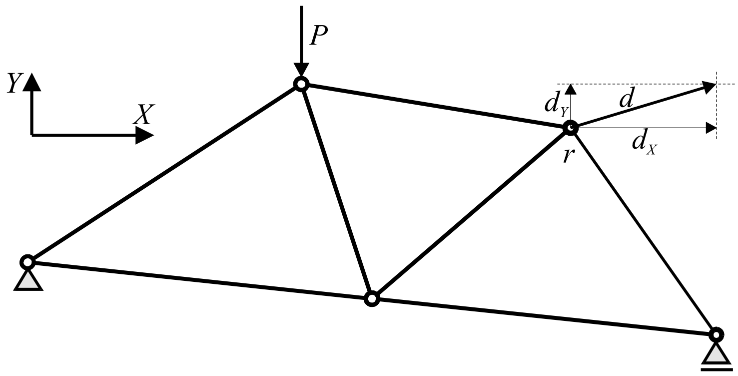


Fig.2.12

Equation (2.72) is like the known equations of the boundary conditions (2.47) and (2.48) but with one difference, here we have obtained nonhomogeneous equations. It changes the procedure of symmetrization of the stiffness matrix. Previously we inserted zeros into suitable columns of the matrix **K** which did not induce any consequences because this matrix was multiplied by zero values of displacements of the support nodes. At this moment we have to keep the components of the matrix occurring in this column because they are multiplied by given displacements (comp. equation (2.72)) and they are usually not equal to zero.

Hence transformations of the stiffness matrix **K** and nodal loads vector **p** leading to the consideration of the geometric load should look as follows:

1. We form vectors $\mathbf{k}_{rX}$ and $\mathbf{k}_{rY}$ which are suitable columns of the matrix **K** joined with the displacements of the node *r*. $\mathbf{k}_{rX}$ is the column with a number equal to the displacement global number u_{rX} and $\mathbf{k}_{rY}$ is the column with a number equal to the displacement global number u_{rY}.
2. We move the nodal forces due to the known displacements d_X and d_Y to the right hand side of the set of equations:

$$\mathbf{p}^d = \mathbf{p} - \mathbf{k}_{rX} d_X - \mathbf{k}_{rY} d_Y . \tag{2.73}$$

3. We consider boundary conditions in a standard way as it was done in Sec.2.6. However, there is one difference, we put known values into the rows of the right hand side vector $\mathbf{p}^d$. These rows have the global numbers equivalent to the degrees of freedom u_{rX} and u_{rY}.

After making the above transformations, the following set of equations rises:

$$\mathbf{K}^r \, \mathbf{u} = \mathbf{p}^{rd} , \tag{2.74}$$

where $\mathbf{K}^r$ is the stiffness matrix which is modified by the standard consideration of the boundary conditions as in equation (2.51) and $\mathbf{p}^{rd}$ is the modified vector $\mathbf{p}^d$ determined by equation (2.73) after inserting given values of displacements:

$P_{rX} = d_X$, $P_{rY} = d_Y$.

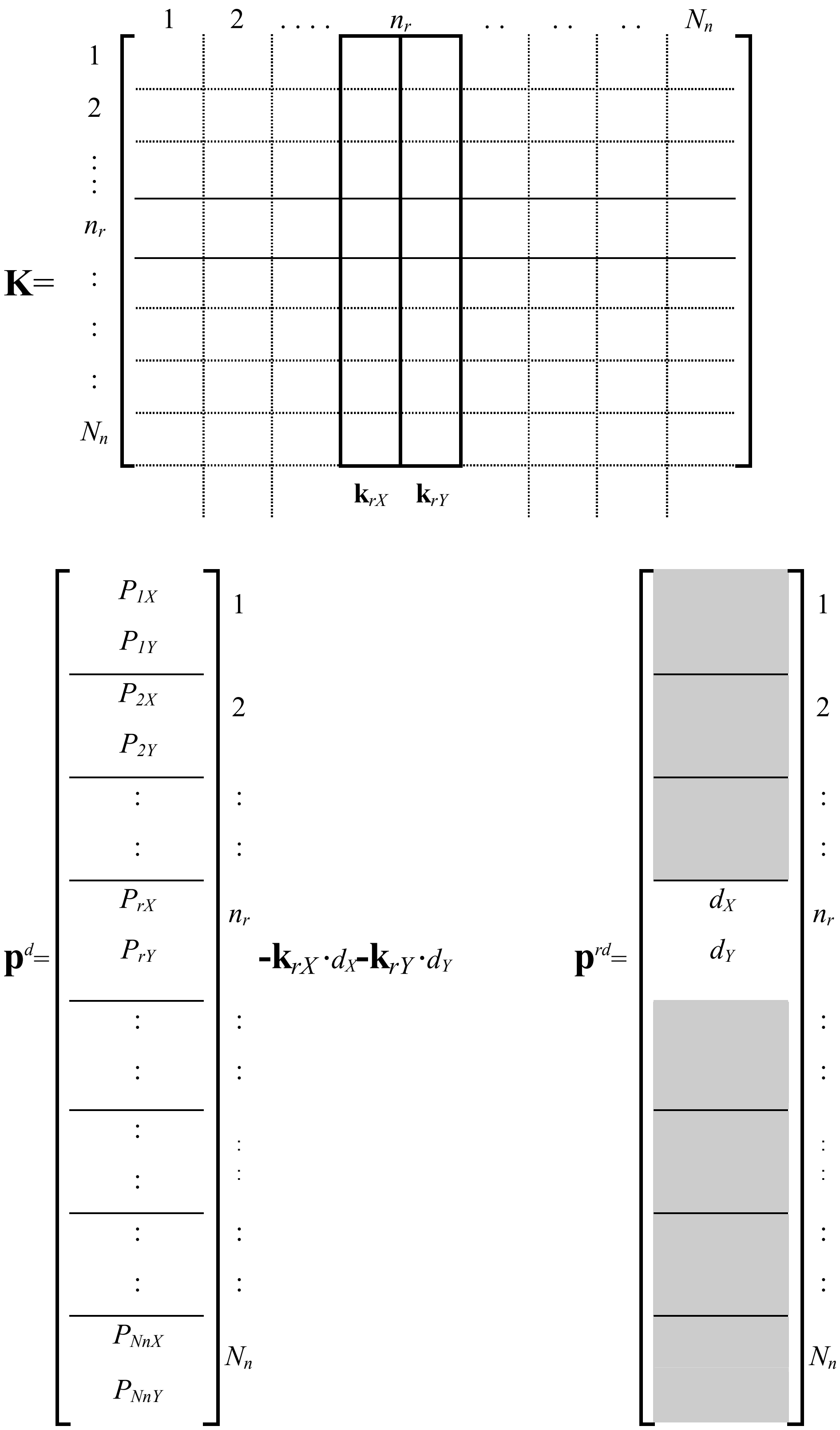


Fig.2.13

2.11. SUPPORT REACTIONS, INTERNAL FORCES AND STRESSES IN ELEMENTS

After aggregation of the stiffness matrix, consideration of the boundary conditions and building the nodal forces vector we obtain the set of linear equations in forms (2.52) or (2.65), or (2.74) with a positively determined symmetric matrix. Methods of solving such equations are described in Appendix 2. The solution of the set of equations is the nodal displacements vector of a structure. Knowing nodal displacements allows us to determine control sums of nodes and support reactions in the support nodes in a very simple way. And then we make use of equation (2.43) in which the matrix **K** does not contain any information about the support constraints.

$$\mathbf{r} = \mathbf{K}\,\mathbf{u} - \mathbf{p}\,. \tag{2.75}$$

The vector of reactions **r** should contain zeros at free nodes and values of reactions at support nodes. If we assume the occurrence of the local coordinate system in some nodes (the „skew" supports), then the obtained components of reactions will be expressed in the local coordinate system.

Since numerical errors resulting from approaching values of numbers stored in the computer memory increase during the solution process, the control sums are rarely equal to zero and they are most often small numbers, for example the order of $1 \cdot 10^{-10}$.

Components of the global displacements vector enable building global displacements vectors of elements (Fig.2.14).

Since the components of the vector **u** are not always written in the global coordinate system (the „skew" supports), then it can happen that some components of the vector $\mathbf{u}^e$ are expressed in the global system and others are expressed in the local coordinate system . To simplify further discussion we standardise the description of the vector bringing down the components to the global coordinate system by taking the advantage of equation (2.57). It should be noted that it is necessary only for elements joined to a node which is supported by a skew support.

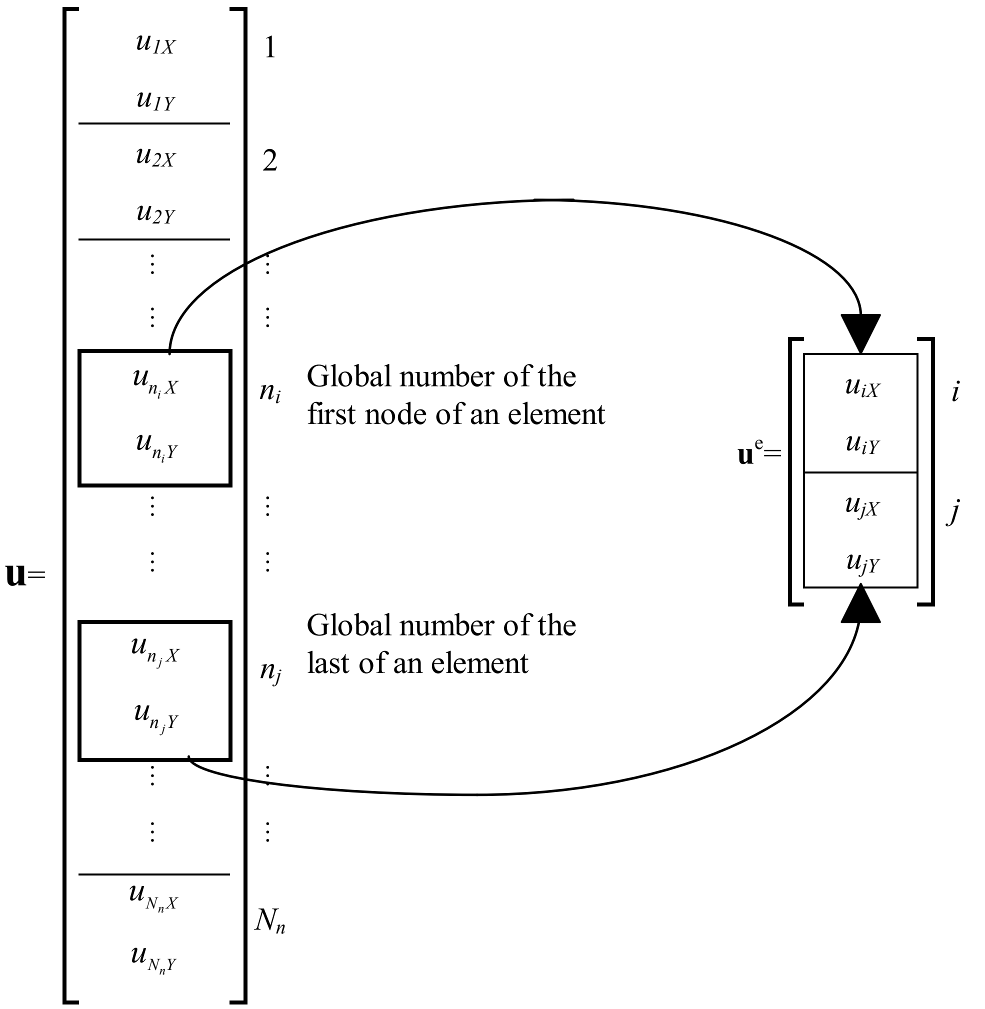


Fig.2.14

Nodal displacements of an element allow to calculate the internal force N in a truss element quite easily. We can either make use of equation (2.11) which requires the knowledge of displacements in the local coordinate system of the element or on the basis of equations (2.9), (2.13) and (2.32) we search the relation:

$$N = \frac{EA}{L}\left[c\left(u_{jX} - u_{iX}\right) + s\left(u_{jY} - u_{iY}\right)\right], \tag{2.76}$$

where similarly to equation (2.18) $c = \cos\alpha$ and $s = \sin\alpha$.

Stresses in the truss element, assuming that the bar is homogeneous, are the axial stresses only which can be calculated using a simple relation:

$$\sigma_x = \frac{N}{A} = \frac{E}{L}\left[c\left(u_{jX} - u_{iX}\right) + s\left(u_{jY} - u_{iY}\right)\right]. \tag{2.77}$$

If the element is loaded with a temperature, then the correction coming from thermal expansion of the material shown in equations (2.76) and (2.77) should be taken into consideration:

$$\sigma_x = E(\varepsilon - \varepsilon_t) = \frac{E}{L}\left[c\left(u_{jX} - u_{iX}\right) + s\left(u_{jY} - u_{iY}\right) - L\left(\alpha_t \Delta t_o\right)\right] \tag{2.78}$$

and

$$N = A\sigma_x = \frac{EA}{L}\left[c\left(u_{jX} - u_{iX}\right) + s\left(u_{jY} - u_{iY}\right) - L\left(\alpha_t \Delta t_o\right)\right]. \tag{2.79}$$

The calculation of displacements, constrained reactions and internal forces in the element finish the static analysis of the truss.