APPENDIX 1
MATRIX ALGEBRA

In this appendix we give the most important definitions of matrix algebra and we elaborate some functions and transformations of matrices which are most helpful in numerical applications and particularly in the finite element method.

A.1.1. DEFINITIONS

- Scalar - value determined only by its magnitude which can be expressed by a real number. The typical scalar values are mass, temperature, time, length, etc. We will denote the scalars by letters written in italic font.
- Vector - value determined by its modulus, direction and sense. The examples of vectors are force, displacement, velocity and rotation. We will denote the vectors by small letters written in bold font.
- Matrix- table containing most often scalars but it can also contain vectors or other matrices. Elements of a matrix are called components. It is a very convenient form of presentation of large quantities of data which we deal with in numerical methods. One of a matrix notation which we apply in this book looks as follows:

$$\mathbf{A} = \left[A_{ij}\right] = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}.$$

  We will denote quadratic matrices (they have the same number of columns and rows) and rectangular matrices (they have a different number of columns and rows) by capital letters written in bold font.
- Column matrix - will also be called a vector and it contains only one column. We will denote it just as vectors.
- Identity matrix - quadratic matrix components of which are equal to zero except for those lying on the main diagonal (diagonal elements). Diagonal elements are equal to 1. We will mark the identity matrix by the capital letter **I** and in some cases by an index pointing dimensions of a matrix:

$$\mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The components of the identity matrix can be written with the help of Kronecker's delta $\mathbf{I} = \left[\delta_{ij}\right]$ where $\delta_{ii} = 1$, $\delta_{ij} = 0$, when $i \neq j$.

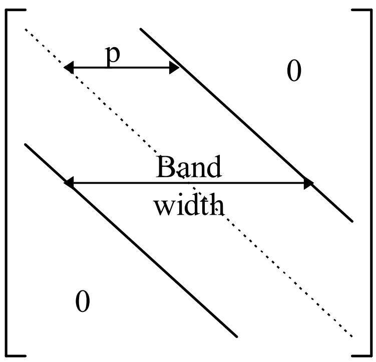
- <u>Triangular matrix</u> - matrix containing either components equal to zero (**L**-triangular lower matrix) lying over the main diagonal or components also equal to zero (**U**- triangular upper matrix) lying below the main diagonal

$$\mathbf{L} = \left[L_{ij}\right] = \begin{bmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix},$$

$$\mathbf{U} = \left[U_{ij}\right] = \begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{bmatrix}.$$

- <u>Band matrix</u> - matrix containing components which are different from zero only when they are close to the main diagonal

$\mathbf{A}_{\text{banded}}$



p - width of half of the band

After suitable grouping of equilibrium equations, stiffness matrices are band matrices in the finite element method.

- <u>Symmetric matrix</u> - matrix with components satisfying the equation:

$$\mathbf{A}_{\text{sym}} \rightarrow \left[A_{ij}\right] = \left[A_{ji}\right]$$

Stiffness matrices are symmetric matrices in the finite element method.

- <u>Transpose matrix</u> - matrix in which we group components so that columns become rows:

$\mathbf{B} = \mathbf{A}^{\mathrm{T}} \rightarrow \left[B_{ij}\right] = \left[A_{ji}\right]$.

Transpose matrices are denoted by the normal capital letter T which is written as an upper index.

- <u>The main diagonal of a matrix</u> is the diagonal which passes from the component A_{11} along other components having equal indices of a column and a row; that is A_{22} ... A_{ii} ... A_{nn}.

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \quad \text{Main diagonal}$$

A.1.2. MATRIX ADDITION AND SUBTRACTION

The operation of matrix addition is defined as follows:

$\mathbf{C} = \mathbf{A} + \mathbf{B} \rightarrow C_{ij} = A_{ij} + B_{ij}$,

which means that the components of the matrix **C** resulting from the addition of matrices **A** and **B** are sums of suitable terms of matrices **A** and **B**. The matrix addition is possible only if both matrices (**A** and **B**) have the same number of columns and rows. The addition is a commutative operation:

$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

Similarly, we define matrix subtraction:

$\mathbf{D} = \mathbf{A} - \mathbf{B} \rightarrow D_{ij} = A_{ij} - B_{ij}$.

Example No 1.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 8 & 2 \\ 2 & 4 & 1 & -2 \\ -1 & 0 & 3 & 4 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 3 & 2 & 5 & 1 \\ 0 & 2 & 1 & 3 \end{bmatrix},$$

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \begin{bmatrix} (1+0) & (3+2) & (8+1) & (2+0) \\ (2+3) & (4+2) & (1+5) & (-2+1) \\ (-1+0) & (0+2) & (3+1) & (4+3) \end{bmatrix} = \begin{bmatrix} 1 & 5 & 9 & 2 \\ 5 & 6 & 6 & -1 \\ -1 & 2 & 4 & 7 \end{bmatrix},$$

$$\mathbf{D} = \mathbf{A} - \mathbf{B} = \begin{bmatrix} (1-0) & (3-2) & (8-1) & (2-0) \\ (2-3) & (4-2) & (1-5) & (-2-1) \\ (-1-0) & (0-2) & (3-1) & (4-3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 7 & 2 \\ -1 & 2 & -4 & -3 \\ -1 & -2 & 2 & 1 \end{bmatrix}.$$

A.1.3. MULTIPLICATION OF A MATRIX BY A SCALAR (SCALING OF A MATRIX)

Scaling a matrix is the name of an operation carried on its components and defined as follows:

$\mathbf{E} = \alpha\mathbf{A} \rightarrow E_{ij} = \alpha A_{ij}$,

which means that components of the matrix **E** resulting from the multiplication of the matrix **A** by the scalar α are products of components of the matrix **A** and the value α.

Example No 2.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 8 & 2 \\ 2 & 4 & 1 & -2 \\ -1 & 0 & 3 & 4 \end{bmatrix}, \qquad \alpha=3.5,$$

$$\mathbf{E} = 3.5\mathbf{A} = \begin{bmatrix} 3.5 & 10.5 & 28.0 & 7.0 \\ 7.0 & 14.0 & 3.5 & -7.0 \\ -3.5 & 0.0 & 10.5 & 14.0 \end{bmatrix}.$$

The matrix **E** which is the result of scaling has the same number of columns and rows just as the matrix **A** does.

A.1.4. MATRIX MULTIPLICATION

Let **C** be the result of multiplication of matrices **A** and **B**:

$\mathbf{C} = \mathbf{A} \times \mathbf{B}$,

then components of the matrix **C** are results of the multiplication of rows of the matrix **A** by columns of the matrix **B** which can be written as follows:

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} ,$$

where n is the number of columns of the matrix **A**. As it is seen the multiplication of the matrices **A** and **B** is possible to perform if the number of columns of the matrix **A** is equal to the number of rows of the matrix **B**. The matrix **C** which is the result of multiplication has the number of rows equal to the number of rows of the matrix **A** and the number of columns equal to the number of columns of the matrix **B**.

$$
\mathbf{C} = \mathbf{A} \times \mathbf{B} =
\begin{array}{c|c}
 & \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1j} & B_{1m} \\ B_{21} & B_{22} & \dots & B_{2j} & B_{2m} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ B_{n1} & B_{n2} & \dots & B_{nj} & B_{nm} \end{bmatrix} \\
\hline
\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ A_{i1} & A_{i2} & \dots & A_{in} \end{bmatrix} & C_{ij}
\end{array}
$$

Example No 3.

$$
\mathbf{A} = \begin{bmatrix} 1 & 3 & 8 & 2 \\ 2 & 4 & 1 & -2 \\ -1 & 0 & 3 & 4 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 3 & 2 & 5 & 1 \\ 0 & 2 & 1 & 3 \end{bmatrix},
$$

$\mathbf{C} = \mathbf{AB}^{\mathrm{T}}$

$\mathbf{AB}^{\mathrm{T}} =$

				0	3	0
				2	2	2
				1	5	1
				0	1	3
1	3	8	2	1·0+3·2+8·1+2·0= =14	1·3+3·2+8·5+2·1= =51	1·0+3·2+8·1+2·3= =20
2	4	1	-2	2·0+4·2+1·1-2 ·0= =9	2·3+4·2+1·5-2·1= =17	2·0+4·2+1·1-2·3= =3
-1	0	3	4	-1·0+0·2+3·1+4·0= =3	-1·3+0·2+3·5+4·1= =16	-1·0+0·2+3·1+4·3= =15

$$
\mathbf{C} = \begin{bmatrix} 14 & 51 & 20 \\ 9 & 17 & 3 \\ 3 & 16 & 15 \end{bmatrix}.
$$

Example No 4.

An interesting result is obtained multiplying a row matrix by a column matrix:

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ -2 \end{bmatrix},$$

$$\mathbf{c} = \mathbf{a}^{\mathrm{T}} \cdot \mathbf{b},$$

$$\mathbf{c} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ -2 \end{bmatrix} = (1 \cdot 3 + 2 \cdot 2 + 3 \cdot 1 + 4 \cdot (-2)) = 2,$$

The matrix **C** with dimensions 1x1 (so it is a scalar) is the result of this operation.

Thus, the vector multiplication $\mathbf{a}^{\mathrm{T}} \cdot \mathbf{b}$ is called scalar multiplication.

The matrix multiplication is not in general the commutative operation which means

$$\mathbf{A}\,\mathbf{B} \neq \mathbf{B}\,\mathbf{A},$$

even if it can be done (it is possible only for quadratic matrices).

We will also give some more definitions concerning matrix multiplication which are worth memorising:

$$(\mathbf{A}\,\mathbf{B})\,\mathbf{C} = \mathbf{A}\,(\mathbf{B}\,\mathbf{C}),$$

$$\mathbf{A}\,(\mathbf{B} + \mathbf{C}) = \mathbf{A}\,\mathbf{B} + \mathbf{A}\,\mathbf{C},$$

$$\mathbf{A}\,\mathbf{I} = \mathbf{I}\,\mathbf{A} = \mathbf{A},$$

$$(\mathbf{A}\,\mathbf{B})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\,\mathbf{A}^{\mathrm{T}}.$$

A.1.5. THE DETERMINANT OF A MATRIX

A determinant is the scalar function of a quadratic matrix which we write as follows:

$$\det \mathbf{A} = \left| A_{ij} \right|.$$

Calculation of the value of a determinant depends on the summation of products obtained from all permutations of components of the matrix **A**:

$$\det \mathbf{A} = \sum_{p} (-1)^{I_p} A_{1\alpha_1}, A_{2\alpha_2}, A_{3\alpha_3}, \ldots A_{n\alpha_n},$$

where p denotes all permutations, I_p - number of inversions in the permutations.

The value of a determinant can also be calculated by using Laplace's expansion with regard to terms of any rows or columns:

$$\det \mathbf{A} = \sum_{k=1}^{n} A_{mk} \overline{A}_{mk} \text{ - development of the row } m \ (1 \le m \le n)$$

or

$$\det \mathbf{A} = \sum_{k=1}^{n} A_{km} \overline{A}_{km} \text{ - development of the column } m \ (1 \le m \le n).$$

$\overline{A}_{ij}$ here signifies the algebraic complement of the element A_{ij} of the matrix:

$$\overline{A}_{ij} = (-1)^{i+j} \left| A_{ij}^* \right|,$$

where $\left| A_{ij}^* \right|$ is the minor of the matrix $\mathbf{A}^* = \left[A_{ij}^* \right]$ that is to say the determinant of a matrix which is obtained by removing the row i and the column j from the matrix **A**.

Laplace's development should be processed as long as we obtain matrices 2x2 whose determinants can be calculated directly:

$$\det \mathbf{A} = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11} A_{22} - A_{12} A_{21}.$$

The way of calculating determinants of the matrix 3x3 (Sarrus's rule) is also known as

$$\det \mathbf{B} = \begin{vmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{vmatrix} =$$

$$= B_{11}B_{22}B_{33} + B_{21}B_{32}B_{13} + B_{31}B_{12}B_{23} - B_{31}B_{22}B_{13} - B_{21}B_{12}B_{33} - B_{11}B_{32}B_{23}.$$

Yet it should not be applied to matrices with a greater number of rows and columns.

It is worth memorising the useful relation:

$$\det(\mathbf{A}\,\mathbf{B}) = \det \mathbf{A} \det \mathbf{B},$$

which helps us to determine determinants of products of matrices effectively.

If the determinant of a matrix is equal to zero, then such a matrix is called a singular matrix.

A.1.6. INVERSE OF A MATRIX

A matrix satisfying the condition:

$$\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$$

is called the inverse of the matrix **A**.

Components of an inverse matrix can be determined by scaling a transpose matrix of algebraic complements:

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \overline{\mathbf{A}}^{\mathrm{T}} = \frac{1}{|A_{ij}|} \left[\overline{A}_{ij} \right]^{\mathrm{T}} = \frac{\left[(-1)^{i+j} \left| A_{ij}^{*} \right| \right]^{\mathrm{T}}}{|A_{ij}|},$$

where $\overline{\mathbf{A}} = \left[\overline{A}_{ij} \right]$ is the matrix of algebraic complements: $\overline{\mathbf{A}} = \left[(-1)^{i+j} \left| A_{ij}^{*} \right| \right]$,

$\left| A_{ij}^{*} \right|$ is the minor, that is, the determinant of a matrix which is formed by removing the row i and the column j from the matrix $\mathbf{A}$.

It is easy to note that it is impossible to find a matrix which would be the 'inverse' of a singular matrix because it requires dividing by zero.

The matrix $\overline{\mathbf{A}}^{\mathrm{T}}$ is called the joined matrix of the matrix $\mathbf{A}$. The joined matrix can be formed for any matrix (even singular).

Example No 5.

We look for the 'inverse' of the matrix:

$$\mathbf{A} = \begin{bmatrix} 9 & 6 & 2 \\ 1 & 9 & 3 \\ 7 & 5 & 3 \end{bmatrix}.$$

First, we calculate the determinant in order to check if the inverse operation is possible. We calculate the determinant of the matrix $\mathbf{A}$ making use of Sarrus's rule:

$$\det \mathbf{A} = (9 \cdot 9 \cdot 3) + (1 \cdot 5 \cdot 2) + (7 \cdot 6 \cdot 3) - (7 \cdot 9 \cdot 2) - (1 \cdot 6 \cdot 3) - (9 \cdot 5 \cdot 3) = 100.$$

We calculate sequencing the algebraic complements:

$$\overline{A}_{11} = (-1)^{1+1} \begin{vmatrix} 9 & 3 \\ 5 & 3 \end{vmatrix} = 12, \qquad \overline{A}_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 3 \\ 7 & 3 \end{vmatrix} = 18,$$

$$\overline{A}_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 9 \\ 7 & 5 \end{vmatrix} = -58, \qquad \overline{A}_{21} = (-1)^{2+1} \begin{vmatrix} 6 & 2 \\ 5 & 3 \end{vmatrix} = -8,$$

$$\overline{A}_{22} = (-1)^{2+2} \begin{vmatrix} 9 & 2 \\ 7 & 3 \end{vmatrix} = 13, \qquad \overline{A}_{23} = (-1)^{2+3} \begin{vmatrix} 9 & 6 \\ 7 & 5 \end{vmatrix} = -3,$$

$$\overline{A}_{31} = (-1)^{3+1} \begin{vmatrix} 6 & 2 \\ 9 & 3 \end{vmatrix} = 0, \qquad \overline{A}_{32} = (-1)^{3+2} \begin{vmatrix} 9 & 2 \\ 1 & 3 \end{vmatrix} = -25,$$

$$\overline{A}_{33} = (-1)^{3+3} \begin{vmatrix} 9 & 6 \\ 1 & 9 \end{vmatrix} = 75,$$

from which we have

$$\mathbf{A}^{-1} = \begin{bmatrix} 0.12 & -0.08 & 0.0 \\ 0.18 & 0.13 & -0.25 \\ -0.58 & -0.03 & 0.75 \end{bmatrix}.$$

A.1.7. DECOMPOSITION OF A MATRIX INTO TRIANGULAR MATRICES

The nonsigular matrix **A** can be broken down into the product of triangular matrices:

$$\mathbf{A} = \mathbf{L}\,\mathbf{U},$$

where **L** is the lower triangular matrix and **U** is the upper triangular matrix. Such a process is called either matrix triangulation or decomposition or factorisation.

The decomposition method was originated by M.H.Doolittle (1878) and later it was reconfirmed by findings of several scientists like Cholesky (ok.1916), A.C.Aitken (1932), T.Banachewicz (1938) and P.D.Crout (1941). The Cholesky method was described by Benoit in 1924.

The components of the triangular matrix **L** and **U** can be calculated using the procedures proposed by Crout or Banachewicz:

$$L_{ii} = 1,\ i = 1 \ldots n,$$

$$U_{ij} = A_{ij} - \sum_{k=1}^{i-1} L_{ik} U_{kj},\ j = i \ldots n,$$

$$L_{ij} = \frac{1}{U_{jj}}\left(A_{ij} - \sum_{k=1}^{j-1} L_{ik} U_{kj} \right),\ i = j \ldots n.$$

Calculation of components is done alternatively for rows of the matrix **U** and columns of the matrix **L** (the Crout method) or in succession the row of the matrix **U** and then the row of the matrix **L** (the Banachewicz method [15]).

Decomposition into triangular matrices is very important in practice because it is applied as the effective method of solving sets of linear equations.

The solution of the set of equations

$$\mathbf{A}\,\mathbf{x} = \mathbf{y}$$

can be obtained in two stages. At the first stage we apply substitutions $\mathbf{A} = \mathbf{L}\,\mathbf{U}$ and $\mathbf{U}\,\mathbf{x} = \mathbf{z}$ which simplify the set of equations to the form:

$$\mathbf{L}\left(\mathbf{U}\,\mathbf{x}\right) = \mathbf{y} \rightarrow \mathbf{L}\,\mathbf{z} = \mathbf{y}$$

which simplifies solving

$$z_1 = \frac{y_1}{L_{11}},$$

$$z_2 = \left(y_2 - L_{21} z_1\right) \frac{1}{L_{22}}, \text{ etc.,}$$

$$z_i = \left(y_i - \sum_{k=1}^{i-1} L_{ik} z_k\right) \frac{1}{L_{ii}}.$$

The applied procedure is called here forward elimination because we calculate consecutively the unknowns z_1, z_2 ... z_i ... z_n.

The second stage depends on the determination of unknown values from equations

$$\mathbf{U}\,\mathbf{x} = \mathbf{z},$$

which is done similarly to the previously used method but we have applied back substitution starting from the last component:

$$x_n = \frac{z_{nn}}{U_{nn}},$$

$$x_{n-1} = \left(z_{n-1} - U_{n-1n} x_n\right) \frac{1}{U_{n-1n-1}}, \text{ etc.,}$$

$$x_i = \left(z_i - \sum_{k=i+1}^{n} U_{ik} x_k\right) \frac{1}{L_{ii}}.$$

Time to solve a set of equations by this method is proportional to $n^3/3$, where n is the number of equations. The number $T_D = n^3/3$ is called the cost of Doolittle's method and is the estimated number of multiplication and division operations which should be done in order to solve a set of equations.

A.1.8. TRIANGULATION OF SYMMETRIC MATRICES

If the quadratic matrix is symmetric (obviously not singular) decomposition given in the previous section can be simplified even more noting that:

$$\mathbf{A} = \mathbf{L}\,\mathbf{L}^{\mathrm{T}} \text{ or } \mathbf{A} = \mathbf{U}^{\mathrm{T}}\,\mathbf{U}.$$

The algorithm of the decomposition of the symmetric matrix **A** into triangular matrices was published for the first time by Cholesky (in 1916) and then independently by Banachewicz (in 1938). This method is usually called the Cholesky method. In Poland the name the Banachewicz-Cholesky method is used in scientific publications.

Components of a triangular lower matrix obtained by this method are equal to:

$L_{ij} = 0$ for $j > i$,

$$L_{ii} = \sqrt{A_{ii} - \sum_{k=1}^{i-1} L_{ik}^2} \ ,$$

$$L_{ij} = \left(A_{ij} - \sum_{k=1}^{j-1} L_{ik} L_{jk} \right) \frac{1}{L_{jj}} \quad \text{for } j < i.$$

In the above equations defining the components lying on the main diagonal of the matrix **L** a square root is applied. The term under the root can certainly be negative and then components of the matrix **L** are complex. It can be proved [5] that for positively defined symmetric matrices the components L_{ii} are always real numbers.

Time of the decomposition of a symmetric matrix obtained by the Banachewicz-Cholesky method is proportional to $T_{B\text{-}CH} = n^3/6$.

Example No 6.

Using the Banachewicz-Cholesky method, find the triangular lower matrix **L** for which $\mathbf{A} = \mathbf{L}\,\mathbf{L}^{\mathrm{T}}$

$$\mathbf{A} = \begin{bmatrix} 10 & 1 & 2 & -1 \\ 1 & 15 & 2 & -3 \\ 2 & 2 & 13 & 4 \\ -1 & -3 & 4 & 12 \end{bmatrix}.$$

We determine particular components of the triangular lower matrix **L** which are different from zero:

$$L_{11} = \sqrt{A_{11}} = \sqrt{10} = 3.16228\,,$$

$$L_{21} = A_{21} \frac{1}{L_{11}} = \frac{1}{\sqrt{10}} = 0.32623\,,$$

$$L_{22} = \sqrt{A_{22} - L_{12}^2} = \sqrt{15 - \left(\frac{1}{\sqrt{10}} \right)^2} = 3.86005\,,$$

$$L_{31} = A_{31} \frac{1}{L_{11}} = \frac{2}{\sqrt{10}} = 0.63246\,,$$

$$L_{32} = \left(A_{32} - L_{31} L_{21} \right) \frac{1}{L_{22}} = \left(2 - \frac{2}{\sqrt{10}} \frac{1}{\sqrt{10}} \right) \frac{1}{\sqrt{14.9}} = 0.46631\,,$$

$$L_{33} = \sqrt{A_{33} - \left(L_{31}^2 + L_{32}^2\right)} = \sqrt{13 - \left(\left(\frac{2}{\sqrt{10}}\right)^2 + \left(\frac{1.8}{\sqrt{14.9}}\right)^2\right)} = 3.51888,$$

$$L_{41} = A_{41}\frac{1}{L_{11}} = -0.31623,$$

$$L_{42} = \left(A_{42} - L_{41}L_{21}\right)\frac{1}{L_{22}} = -0.75129,$$

$$L_{43} = \left(A_{43} - \left(L_{41}L_{31} + L_{42}L_{32}\right)\right)\frac{1}{L_{33}} = 1.29312,$$

$$L_{44} = \sqrt{A_{44} - \left(L_{41}^2 + L_{42}^2 + L_{43}^2\right)} = 3.10860.$$

$$\mathbf{L} = \begin{bmatrix} 3.16228 & 0 & 0 & 0 \\ 0.31623 & 3.86005 & 0 & 0 \\ 0.63246 & 0.46631 & 3.51888 & 0 \\ -0.31623 & -0.75129 & 1.29312 & 3.10860 \end{bmatrix}$$

A.1.9. ORTHOGONAL MATRICES

There is a group of matrices having the property:

$$\mathbf{A}^{-1} = \mathbf{A}^{\mathrm{T}}$$

which enormously simplifies solving a set of equations. We say that such matrices are orthogonal matrices. This property is shown by the transformation matrices for vectors:

$$\mathbf{R} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix},$$

where $c = \cos\alpha$, $s = \sin\alpha$, and α is a rotation angle.

We check the orthogonality of this matrix by the equation $\mathbf{R}\,\mathbf{R}^{\mathrm{T}} = \mathbf{I}$:

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \times \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} c^2 + s^2 & cs - sc \\ sc - cs & c^2 + s^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We use this property of the transformation matrix in some chapters of this book.