Example 3.E1.

The content of the example

A 3D truss made from a material with Young's modulus $E=2\cdot10^8$ kPa possessing the same pipe cross sections for all elements is loaded with the concentrated force P=12 kN. The structure scheme and dimensions of cross sections are given in Fig.3.E1.1. Nodal coordinates and components of internal forces in the global coordinate system are given in Tab.3.E1.1.

Calculate nodal displacements and internal forces in the elements of the truss.

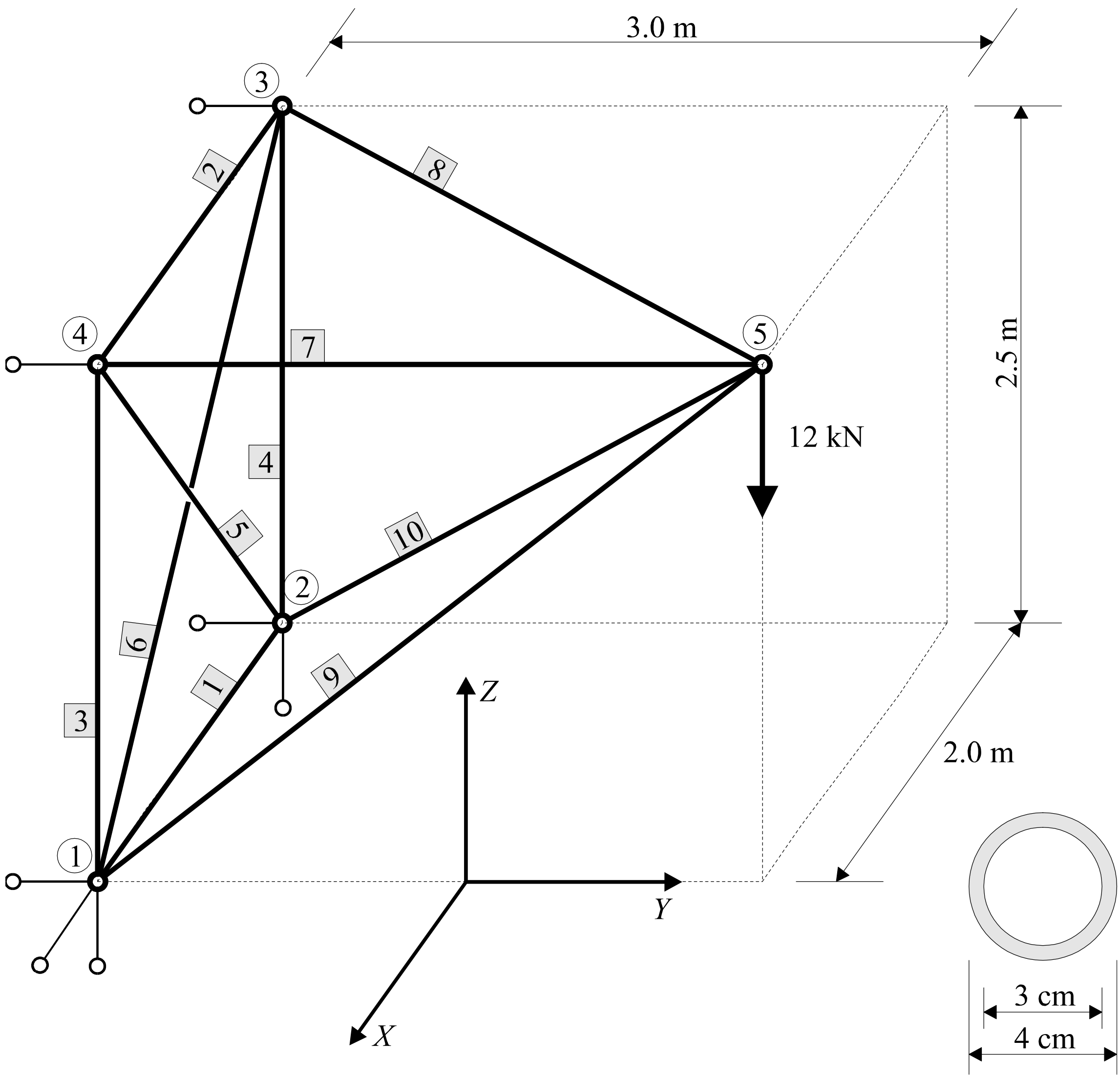

Fig.3.E1.1

Tab.3.E1.1

	Coordinates			Internal forces		
Node No n	X_n [m]	Y_n [m]	Z_n [m]	P_{Xn} [kN]	P_{Yn} [kN]	P_{Zn} [kN]
1	2.0	0.0	0.0	------	------	------
2	0.0	0.0	0.0	------	------	------
3	0.0	0.0	2.5	------	------	------
4	2.0	0.0	2.5	------	------	------
5	2.0	3.0	2.5	------	------	-12

The solution of the example

We start solving the problem from calculating the area of cross sections

$A=((4.0\text{ cm})^2-(3.0\text{ cm})^2)\ \pi/4=5.4978\text{ cm}^2$.

Next we determine projection of elements on axes of the global coordinate system and we calculate direction cosines. The data are collected in Tab.3.E1.1 which also contains length of the elements (the column L_n).

Tab.3.Z1.1

							Direction cosines		
Elem. No n	Node No n_i	 n_j	L_{nX} [m]	L_{nY} [m]	L_{nZ} [m]	L_n [m]	c_X	c_Y	c_Z
1	2	1	2.0	0.0	0.0	2.0	1.0	0.0	0.0
2	3	4	2.0	0.0	0.0	2.0	1.0	0.0	0.0
3	1	4	0.0	0.0	2.5	2.5	0.0	0.0	1.0
4	2	3	0.0	0.0	2.5	2.5	0.0	0.0	1.0
5	2	4	2.0	0.0	2.5	3.20156	0.62470	0.0	0.78087
6	1	3	-2.0	0.0	2.5	3.20156	-0.62470	0.0	0.78087
7	4	5	0.0	3.0	0.0	3.0	0.0	1.0	0.0
8	3	5	2.0	3.0	0.0	3.60555	0.55470	0.83205	0.0
9	1	5	0.0	3.0	2.5	3.90512	0.0	0.76822	0.64018
10	2	5	2.0	3.0	2.5	4.38748	0.45584	0.68376	0.56980

Similarly to example 2.E1 we form the nodal force vector **p** inserting the suitable components of forces directly into the suitable rows of this vector. Since no forces act on support nodes, then the vector $\mathbf{p}^r$ considering boundary conditions is identical to the vector **p**.

$$\mathbf{p}=\begin{array}{c}\left[\begin{array}{c} 0.0 \\ 0.0 \\ 0.0 \\ \hline 0.0 \\ 0.0 \\ 0.0 \\ \hline 0.0 \\ 0.0 \\ 0.0 \\ \hline 0.0 \\ 0.0 \\ 0.0 \\ \hline 0.0 \\ 0.0 \\ -12.0 \end{array}\right]\end{array}\begin{array}{c} \\ 1 \\ \\ \\ 2 \\ \\ \\ 3 \\ \\ \\ 4 \\ \\ \\ 5 \\ \\ \end{array} \qquad \mathbf{p}=\mathbf{p}^r$$

We calculate the blocks $\mathbf{J}^e$ of the element stiffness matrix of the truss according to equation (3.18).

$$\mathbf{J}^1=\begin{bmatrix} 54978.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} \qquad \mathbf{J}^2=\begin{bmatrix} 54978.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$\mathbf{J}^3=\begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 43982.4 \end{bmatrix} \qquad \mathbf{J}^4=\begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 43982.4 \end{bmatrix}$$

$$\mathbf{J}^5=\begin{bmatrix} 13402.7 & 0.0 & 16753.4 \\ 0.0 & 0.0 & 0.0 \\ 16753.4 & 0.0 & 20941.8 \end{bmatrix} \qquad \mathbf{J}^6=\begin{bmatrix} 13402.7 & 0.0 & -16753.4 \\ 0.0 & 0.0 & 0.0 \\ -16753.4 & 0.0 & 20941.8 \end{bmatrix}$$

$$\mathbf{J}^7=\begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 36652.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} \qquad \mathbf{J}^8=\begin{bmatrix} 9383.48 & 14075.2 & 0.0 \\ 14075.2 & 21112.8 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$\mathbf{J}^9=\begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 16617.2 & 13847.6 \\ 0.0 & 13847.6 & 11539.7 \end{bmatrix} \qquad \mathbf{J}^{10}=\begin{bmatrix} 5207.54 & 7811.31 & 6509.43 \\ 7811.31 & 11717.0 & 9764.14 \\ 6509.43 & 9764.14 & 8136.79 \end{bmatrix}$$

After aggregation of the stiffness matrix, we obtain the global matrix **K** expressed by the blocks $\mathbf{J}^e$:

$$\mathbf{K}=\begin{array}{cc}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} & \\
\left[\begin{array}{c|c|c|c|c}
\mathbf{J}^1+\mathbf{J}^3+\mathbf{J}^6+\mathbf{J}^9 & -\mathbf{J}^1 & -\mathbf{J}^6 & -\mathbf{J}^3 & -\mathbf{J}^9 \\ \hline
-\mathbf{J}^1 & \mathbf{J}^1+\mathbf{J}^4+\mathbf{J}^5+\mathbf{J}^{10} & -\mathbf{J}^4 & -\mathbf{J}^5 & -\mathbf{J}^{10} \\ \hline
-\mathbf{J}^6 & -\mathbf{J}^4 & \mathbf{J}^2+\mathbf{J}^4+\mathbf{J}^6+\mathbf{J}^8 & -\mathbf{J}^2 & -\mathbf{J}^8 \\ \hline
-\mathbf{J}^3 & -\mathbf{J}^5 & -\mathbf{J}^2 & \mathbf{J}^2+\mathbf{J}^3+\mathbf{J}^5+\mathbf{J}^7 & -\mathbf{J}^7 \\ \hline
-\mathbf{J}^9 & -\mathbf{J}^{10} & -\mathbf{J}^8 & -\mathbf{J}^7 & \mathbf{J}^7+\mathbf{J}^8+\mathbf{J}^9+\mathbf{J}^{10}
\end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\end{array}$$

or after inserting the values of components into the above matrix, we obtain a new version of the matrix **K** in the form presented on the next page.

The following equations come from boundary conditions:

(1) $u_{1X}=0$, (2) $u_{1Y}=0$, (3) $u_{1Z}=0$, (5) $u_{2Y}=0$, (6) $u_{2Z}=0$, (8) $u_{3Y}=0$, (11) $u_{4Y}=0$,
where the global number of degrees of freedom are given in brackets.

Similarly to the example 2.E1 we consider the boundary conditions by inserting zeros into suitable rows and columns of the matrix **K** and inserting the values 1.0 into the diagonals in the rows with numbers in accordance with the global numbers of degrees of freedom of constraints. The matrix $\mathbf{K}^r$ containing the modified value (marked in italic fonts) is shown on the following page after the matrix **K**.

The global stiffness matrix:

$$
\begin{array}{cc}
\begin{array}{ccccccccccccccc}
 & 1 & & & 2 & & & 3 & & & 4 & & & 5 &
\end{array} & \\
\mathbf{K}=\left[\begin{array}{ccc|ccc|ccc|ccc|ccc}
68380.7 & 0 & -16753.4 & -54978.0 & 0 & 0 & -13402.7 & 0 & 16753.4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 16617.2 & 13847.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -16617.2 & -13847.6 \\
-16753.4 & 13847.6 & 76463.8 & 0 & 0 & 0 & 16753.4 & 0 & -20941.8 & 0 & 0 & -43982.4 & 0 & -13847.6 & -11539.7 \\
\hline
-54978.0 & 0 & 0 & 73588.3 & 7811.3 & 23262.8 & 0 & 0 & 0 & -13402.7 & 0 & -16753.4 & -5207.5 & -7811.3 & -6509.4 \\
0 & 0 & 0 & 7811.3 & 11717.0 & 9764.1 & 0 & 0 & 0 & 0 & 0 & 0 & -7811.3 & -11717.0 & -9764.1 \\
0 & 0 & 0 & 23262.8 & 9764.1 & 73060.9 & 0 & 0 & -43982.4 & -16753.4 & 0 & -20941.8 & -6509.4 & -9764.1 & -8136.8 \\
\hline
-13402.7 & 0 & 16753.4 & 0 & 0 & 0 & 77764.2 & 14075.2 & -16753.4 & -54978.0 & 0 & 0 & -9383.48 & -14075.2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 14075.2 & 21112.8 & 0 & 0 & 0 & 0 & -14075.2 & -21112.8 & 0 \\
16753.4 & 0 & -20941.8 & 0 & 0 & -43982.4 & -16753.4 & 0 & 64924.2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -13402.7 & 0 & -16753.4 & -54978.0 & 0 & 0 & 68380.7 & 0 & 16753.4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 36652.0 & 0 & 0 & -36652.0 & 0 \\
0 & 0 & -43982.4 & -16753.4 & 0 & -20941.8 & 0 & 0 & 0 & 16753.4 & 0 & 64924.2 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -5207.5 & -7811.3 & -6509.4 & -9383.5 & -14075.2 & 0 & 0 & 0 & 0 & 14591.0 & 21886.5 & 6509.4 \\
0 & -16617.2 & -13847.6 & -7811.3 & -11717.0 & -9764.1 & -14075.2 & -21112.8 & 0 & 0 & -36652.0 & 0 & 21886.5 & 86099.0 & 23611.8 \\
0 & -13847.6 & -11539.7 & -6509.4 & -9764.1 & -8136.8 & 0 & 0 & 0 & 0 & 0 & 0 & 6509.4 & 23611.8 & 19676.5
\end{array}\right] &
\begin{array}{c}
\\ 1 \\ \\ \\ 2 \\ \\ \\ 3 \\ \\ \\ 4 \\ \\ \\ 5 \\ \\
\end{array}
\end{array}
$$

The global stiffness matrix after taking into consideration boundary conditions:

$$
\mathbf{K}^r = \begin{array}{cc}
\begin{array}{ccccccccccccccc} & 1 & & & 2 & & & 3 & & & 4 & & & 5 & \end{array} & \\
\left[\begin{array}{ccc|ccc|ccc|ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 73588.3 & 0 & 0 & 0 & 0 & 0 & -13402.7 & 0 & -16753.4 & -5207.5 & -7811.3 & -6509.4 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 77764.2 & 0 & -16753.4 & -54978.0 & 0 & 0 & -9383.48 & -14075.2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -16753.4 & 0 & 64924.2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -13402.7 & 0 & 0 & -54978.0 & 0 & 0 & 68380.7 & 0 & 16753.4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -16753.4 & 0 & 0 & 0 & 0 & 0 & 16753.4 & 0 & 64924.2 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -5207.5 & 0 & 0 & -9383.5 & 0 & 0 & 0 & 0 & 0 & 14591.0 & 21886.5 & 6509.4 \\
0 & 0 & 0 & -7811.3 & 0 & 0 & -14075.2 & 0 & 0 & 0 & 0 & 0 & 21886.5 & 86099.0 & 23611.8 \\
0 & 0 & 0 & -6509.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6509.4 & 23611.8 & 19676.5
\end{array}\right] & \begin{array}{c} \\ 1 \\ \\ \\ 2 \\ \\ \\ 3 \\ \\ \\ 4 \\ \\ \\ 5 \\ \\ \end{array}
\end{array}
$$

The matrix $\mathbf{K}^r$ can be applied in practice as the matrix of the FEM set of equations

$\mathbf{K}^r\,\mathbf{u} = \mathbf{p}^r$.

After solving the above set, we obtain the nodal displacement vector **u** and according to equation (2.75) the support reaction vector **r**

$$\mathbf{u}=\left[\begin{array}{c}0.0\\0.0\\0.0\\\hline -0.30705E\text{-}4\\0.0\\0.0\\\hline 0.18593E\text{-}3\\0.0\\0.47977E\text{-}4\\\hline 0.15522E\text{-}3\\0.0\\-0.47977E\text{-}4\\\hline 0.16367E\text{-}3\\0.25471E\text{-}3\\-0.97982E\text{-}3\end{array}\right]\begin{array}{c}\\1\\\\\\2\\\\\\3\\\\\\4\\\\\\5\\\\\end{array}\qquad\qquad\mathbf{r}=\left[\begin{array}{c}0.0\\9.336\\12.0\\\hline \text{------}\\5.064\\0.0\\\hline \text{------}\\-5.064\\\text{------}\\\hline \text{------}\\-9.336\\\text{------}\\\hline \text{------}\\\text{------}\\\text{------}\end{array}\right]\begin{array}{c}\\1\\\\\\2\\\\\\3\\\\\\4\\\\\\5\\\\\end{array}$$

Tab.3.E1.2 contains nodal displacements and internal forces in the elements calculated according to equation (3.32).

Tab.3.E1.2

We obtain columns marked by u_{iX}, u_{iY}, u_{jX}, u_{jY} in the way shown in Fig.2.14 (Chapter II)

No n	u_{iX} [m]	u_{iY} [m]	u_{iZ} [m]	u_{jX} [m]	u_{jY} [m]	u_{jZ} [m]	N [kN] (3.32)
1	-0.30705E-4	0.0	0.0	0.0	0.0	0.0	1.688
2	0.18593E-3	0.0	0.47977E-4	0.15522E-3	0.0	-0.47977E-4	-1.688
3	0.0	0.0	0.0	0.15522E-3	0.0	-0.47977E-4	-2.110
4	-0.30705E-4	0.0	0.0	0.18593E-3	0.0	0.47977E-4	2.110
5	-0.30705E-4	0.0	0.0	0.15522E-3	0.0	-0.47977E-4	2.702
6	0.0	0.0	0.0	0.18593E-3	0.0	0.47977E-4	-2.702
7	0.15522E-3	0.0	-0.47977E-4	0.16367E-3	0.25471E-3	-0.97982E-3	9.336
8	0.18593E-3	0.0	0.47977E-4	0.16367E-3	0.25471E-3	-0.97982E-3	6.087
9	0.0	0.0	0.0	0.16367E-3	0.25471E-3	-0.97982E-3	-12.152
10	-0.30705E-4	0.0	0.0	0.16367E-3	0.25471E-3	-0.97982E-3	-7.407

Fig.3.E1.1 shows the scheme of the structure deformation.

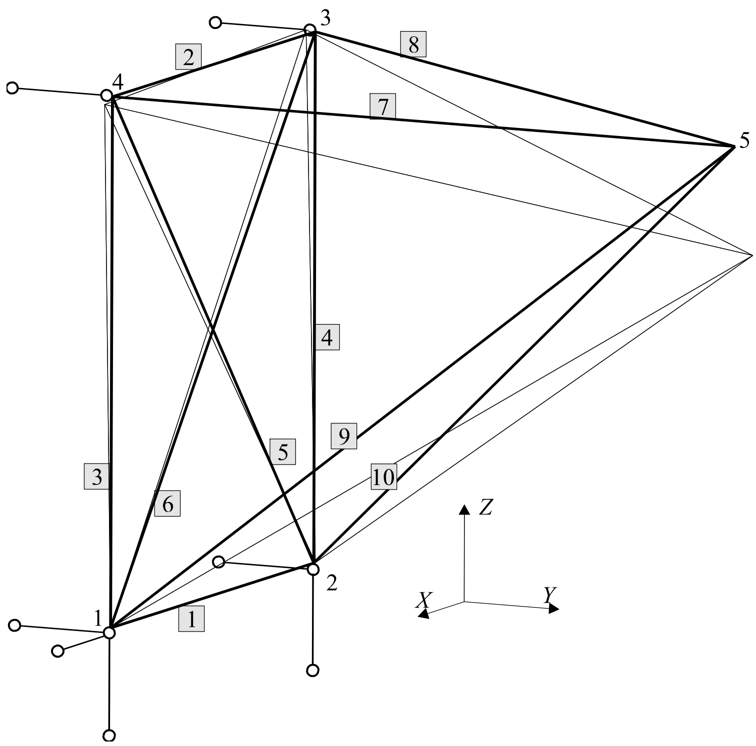

Fig.3.E1.2