Example 4.E2.

The content of the example

A 2D frame with the static scheme shown in Fig.4.E2.1 and made from a material with $E=1.5 \cdot 10^7$ kPa is loaded with static (a concentrated moment at node No 2) and temperature loads. A moveable support of node No 5 is located in such a way that its motion is not parallel to any axes of the global coordinate system. The local coordinate system $x'y'$ is rotated with regard to the global system XY by the angle of 10°.

Determine nodal displacements of the frame (u_{iX}, u_{iY}, φ_i) and find the distribution of internal forces.

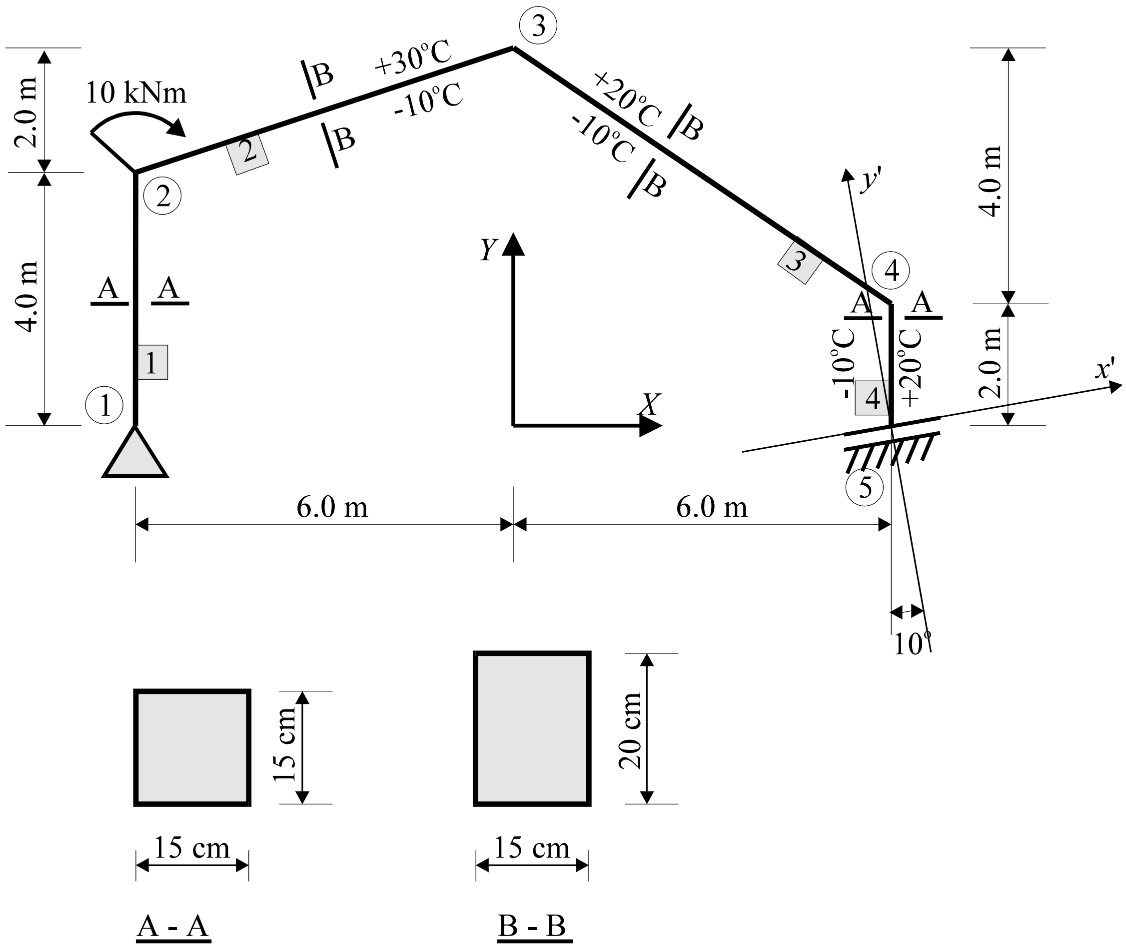

Fig.4.E2.1

The solution of the example

We start solving the problem from grouping the necessary data. The nodal coordinates of the frame and the values of nodal loads are shown in Tab.4.E2.1, and the global numbers of the first and last nodes of elements, the lengths of elements and geometric characteristics are presented in Tab.4.E2.2 where A means the area of a cross section and J_z is the moment of inertia of a cross section with regard to the z axis of the local system (or Z of the global system because axes Z and z cover each other).

Tab.4.E2.1

Node No n	X_n [m]	Y_n [m]	P_{Xn} [kN]	P_{Yn} [kN]	M_n [kNm]
1	-6.0	0.0	------	------	------
2	-6.0	4.0	------	------	-10
3	0.0	6.0	------	------	------
4	6.0	2.0	------	------	------
5	6.0	0.0	------	------	------

Tab.4.E2.2

No n	Node No n_i	n_j	L_{nX} [m]	L_{nY} [m]	L_n [m]	α_n [rad]	Cross section	A [m^2]	J_z [m^4]
1	1	2	0	4	4.0	1.570796	A-A	$2.25 \cdot 10^{-2}$	$4218.75 \cdot 10^{-8}$
2	2	3	6	2	6.3246	0.321750	B-B	$3.0 \cdot 10^{-2}$	$1.0 \cdot 10^{-4}$
3	4	3	-6	4	7.2111	2.553590	B-B	$3.0 \cdot 10^{-2}$	$1.0 \cdot 10^{-4}$
4	5	4	0	2	2.0	1.570796	A-A	$2.25 \cdot 10^{-2}$	$4218.75 \cdot 10^{-8}$

Next we will determine nodal force vectors caused by external loads. We use equation (4.72) in order to calculate components of the vectors $\mathbf{f}'^2$, $\mathbf{f}'^3$, $\mathbf{f}'^4$ in the local system and then we transform them to the global system ($\mathbf{f}^2$, $\mathbf{f}^3$, $\mathbf{f}^4$) with the help of equations (2.31) and (4.26)

$$\mathbf{f}'^2 = \left[\begin{array}{c} 45.0 \\ 0.0 \\ -3.0 \\ \hline -45.0 \\ 0.0 \\ 3.0 \end{array}\right] \begin{array}{c} 2 \\ \\ \\ 3 \\ \\ \\ \end{array} \qquad \mathbf{f}^2 = \left[\begin{array}{c} 42.6907 \\ 14.2302 \\ -3.0 \\ \hline -42.6907 \\ -14.2302 \\ 3.0 \end{array}\right] \begin{array}{c} 2 \\ \\ \\ 3 \\ \\ \\ \end{array}$$

$$\mathbf{f}'^3 = \left[\begin{array}{c} 22.5 \\ 0.0 \\ 2.25 \\ \hline -22.5 \\ 0.0 \\ -2.25 \end{array}\right] \begin{array}{c} 4 \\ \\ \\ 3 \\ \\ \\ \end{array} \qquad \mathbf{f}^3 = \left[\begin{array}{c} -18.7211 \\ 12.4808 \\ 2.25 \\ \hline 18.7211 \\ -12.4808 \\ -2.25 \end{array}\right] \begin{array}{c} 4 \\ \\ \\ 3 \\ \\ \\ \end{array}$$

$$\mathbf{f}'^4 = \left[\begin{array}{c} 16.875 \\ 0.0 \\ 1.2656 \\ \hline -16.875 \\ 0.0 \\ -1.2656 \end{array}\right] \begin{array}{c} 5 \\ \\ \\ 4 \\ \\ \\ \end{array} \qquad \mathbf{f}^4 = \left[\begin{array}{c} 0.0 \\ 16.875 \\ 1.2656 \\ \hline 0.0 \\ -16.875 \\ -1.2656 \end{array}\right] \begin{array}{c} 5 \\ \\ \\ 4 \\ \\ \\ \end{array}$$

We also calculate the stiffness matrices ($\mathbf{K}^1, \mathbf{K}^2, \mathbf{K}^3, \mathbf{K}^4$) of frame elements using equation (4.27):

$$\mathbf{K}^1 = \begin{array}{cc} \begin{array}{cc} \qquad\qquad 1 \qquad\qquad & \qquad\qquad 2 \qquad\qquad \end{array} & \\ \left[\begin{array}{ccc|ccc} 118.65 & 0.0 & -237.31 & -118.65 & 0.0 & -237.31 \\ 0.0 & 84375.0 & 0.0 & 0.0 & -84375.0 & 0.0 \\ -237.31 & 0.0 & 632.82 & 237.31 & 0.0 & 316.41 \\ \hline -118.65 & 0.0 & 237.31 & 118.65 & 0.0 & 237.31 \\ 0.0 & -84375.0 & 0.0 & 0.0 & 84375.0 & 0.0 \\ -237.31 & 0.0 & 316.41 & 237.31 & 0.0 & 632.82 \end{array}\right] & \begin{array}{c} \\ 1 \\ \\ \\ 2 \\ \\ \end{array} \end{array}$$

$$\mathbf{K}^2 = \begin{array}{cc} \begin{array}{cc} \qquad\qquad 2 \qquad\qquad & \qquad\qquad 3 \qquad\qquad \end{array} & \\ \left[\begin{array}{ccc|ccc} 64043.2 & 21324.0 & -71.15 & -64043.2 & -21324.0 & -71.15 \\ 21324.0 & 7179.16 & 213.45 & -21324.0 & -7179.16 & 213.45 \\ -71.15 & 213.45 & 948.68 & 71.15 & -213.45 & 474.34 \\ \hline -64043.2 & -21324.0 & 71.15 & 64043.2 & 21324.0 & 71.15 \\ -21324.0 & -7179.16 & -213.45 & 21324.0 & 7179.16 & -213.45 \\ -71.15 & 213.45 & 474.34 & 71.15 & -213.45 & 948.68 \end{array}\right] & \begin{array}{c} \\ 2 \\ \\ \\ 3 \\ \\ \end{array} \end{array}$$

$$\mathbf{K}^3 = \begin{array}{cc} \begin{array}{cc} \qquad\qquad 4 \qquad\qquad & \qquad\qquad 3 \qquad\qquad \end{array} & \\ \left[\begin{array}{ccc|ccc} 43217.4 & -28779.6 & -96.01 & -43217.4 & 28779.6 & -96.01 \\ -28779.6 & 19234.4 & -144.01 & 28779.6 & -19234.4 & -144.01 \\ -96.01 & -144.01 & 832.05 & 96.01 & 144.01 & 416.03 \\ \hline -43217.4 & 28779.6 & 96.01 & 43217.4 & -28779.6 & 96.01 \\ 28779.6 & -19234.4 & 144.01 & -28779.6 & 19234.4 & 144.01 \\ -96.01 & -144.009 & 416.03 & 96.01 & 144.01 & 832.05 \end{array}\right] & \begin{array}{c} \\ 4 \\ \\ \\ 3 \\ \\ \end{array} \end{array}$$

$$\mathbf{K}^4 = \begin{array}{cc} \begin{array}{cc} \qquad\qquad 5 \qquad\qquad & \qquad\qquad 4 \qquad\qquad \end{array} & \\ \left[\begin{array}{ccc|ccc} 949.23 & 0.0 & -949.23 & -949.23 & 0.0 & -949.23 \\ 0.0 & 168750.0 & 0.0 & 0.0 & -168750.0 & 0.0 \\ -949.23 & 0.0 & 1265.64 & 949.23 & 0.0 & 632.82 \\ \hline -949.23 & 0.0 & 949.23 & 949.23 & 0.0 & 949.23 \\ 0.0 & -168750. & 0.0 & 0.0 & 168750.0 & 0.0 \\ -949.23 & 0.0 & 632.82 & 949.23 & 0.0 & 1265.64 \end{array}\right] & \begin{array}{c} \\ 5 \\ \\ \\ 4 \\ \\ \end{array} \end{array}$$

The components of the above matrices are defined in the global coordinate system.

Now we have to transform both the load vector and the stiffness matrix of element No 4 touching the „skew" support at node No 5 to the local system $x'y'$. The element transformation matrix $\mathbf{R}'^4$ is composed of the identity matrix $\mathbf{I}$ and the nodal rotation matrix $\mathbf{R}'_5$:

$$\mathbf{R}'^4 = \begin{bmatrix} \mathbf{R}'_5 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

where

$$\mathbf{R}'_5 = \begin{bmatrix} \cos(10^o) & -\sin(10^o) & 0 \\ \sin(10^o) & \cos(10^o) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.984808 & -0.173648 & 0 \\ 0.173648 & 0.984808 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and after inserting proper values we obtain

$$\mathbf{R}'^4 = \left[\begin{array}{ccc|ccc} 0.984808 & -0.173648 & 0 & 0 & 0 & 0 \\ 0.173648 & 0.984808 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

The transformation of the nodal force vector $\mathbf{f}^4$ to the local system $x'y'$ gives

$$\mathbf{f}'^4_5 = \left(\mathbf{R}'^4\right)^T \mathbf{f}^4 = \begin{array}{c} \left[\begin{array}{c} 2.9303 \\ 16.6186 \\ 1.2656 \\ \hline 0.0 \\ -16.875 \\ -1.2656 \end{array}\right] \end{array} \begin{array}{c} 5 \\ \\ \\ \\ 4 \end{array} .$$

The transformation of the stiffness matrix for element No 4 done according to equation (2.60) leads to the relation

$$
\mathbf{K}'^{4} = \left(\mathbf{R}'^{4}\right)^{\mathrm{T}} \mathbf{K}^{4} \mathbf{R}'^{4} = \left[\begin{array}{ccc|ccc}
6009.03 & 28695.6 & -934.81 & -934.81 & -29303.1 & -934.81 \\
28695.6 & 163690.0 & 164.83 & 164.83 & -166186.0 & 164.83 \\
-934.81 & 164.83 & 1265.64 & 949.23 & 0.0 & 632.82 \\
\hline
-934.81 & 164.83 & 949.23 & 949.23 & 0.0 & 949.23 \\
-29303.1 & -166186.0 & 0.0 & 0.0 & 168750.0 & 0.0 \\
-934.81 & 164.83 & 632.82 & 949.23 & .0.0 & 1265.64
\end{array}\right]
$$

$$
\mathbf{K}'^{4} = \begin{bmatrix} \mathbf{K}'^{4}_{55} & \mathbf{K}'^{4}_{54} \\ \mathbf{K}'^{4}_{45} & \mathbf{K}'^{4}_{44} \end{bmatrix}
$$

We can now juxtapose the global vector of nodal forces for the frame:

$$
\mathbf{p} = \left[\begin{array}{c}
0.0 \\ 0.0 \\ 0.0 \\ \hline 0.0 \\ 0.0 \\ -10.0 \\ \hline 0.0 \\ 0.0 \\ 0.0 \\ \hline 0.0 \\ 0.0 \\ 0.0 \\ \hline 0.0 \\ 0.0 \\ 0.0
\end{array}\right] - \left[\begin{array}{c}
\\ \\ \\ \hline 42.6907 \\ 14.2302 \\ -3.0 \\ \hline -42.6907 \\ -14.2302 \\ 3.0 \\ \hline \\ \\ \\ \hline \\ \\ \\
\end{array}\right] - \left[\begin{array}{c}
\\ \\ \\ \hline \\ \\ \\ \hline 18.7211 \\ -12.4808 \\ -2.25 \\ \hline -18.7211 \\ 12.4808 \\ 2.25 \\ \hline \\ \\ \\
\end{array}\right] - \left[\begin{array}{c}
\\ \\ \\ \hline \\ \\ \\ \hline \\ \\ \\ \hline 0.0 \\ -16.875 \\ -1.2656 \\ \hline 2.9303 \\ 16.6186 \\ 1.2656
\end{array}\right] = \left[\begin{array}{c}
0.0 \\ 0.0 \\ 0.0 \\ \hline -42.6907 \\ -14.2302 \\ -7.0 \\ \hline 23.9696 \\ 26.711 \\ -0.75 \\ \hline 18.7211 \\ 4.3942 \\ -0.9844 \\ \hline -2.9303 \\ -16.6186 \\ -1.2656
\end{array}\right]
\begin{array}{c}
\\ 1 \\ \\ \\ 2 \\ \\ \\ 3 \\ \\ \\ 4 \\ \\ \\ 5 \\ \\
\end{array}
$$

After taking into consideration boundary conditions at nodes No 1 and No 5

$u_1 = 0$, $v_1 = 0$, $\varphi_1 = 0$, $v'_5 = 0$, $\varphi_5 = 0$, we obtain

$$\mathbf{p}^r = \left[\begin{array}{c} \mathit{0} \\ \mathit{0} \\ 0.0 \\ \hline -42.6907 \\ -14.2302 \\ -7.0 \\ \hline 23.9696 \\ 26.711 \\ -0.75 \\ \hline 18.7211 \\ 4.3942 \\ -0.9844 \\ \hline -2.9303 \\ \mathit{0} \\ \mathit{0} \end{array}\right] \begin{array}{c} \\ 1 \\ \\ \\ 2 \\ \\ \\ 3 \\ \\ \\ 4 \\ \\ \\ 5 \\ \\ \end{array} ,$$

The aggregation of the global stiffness matrix can be written by the equation

$$\mathbf{K} = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{c|c|c|c|c} \mathbf{K}_{11}^{1} & \mathbf{K}_{12}^{1} & & & \\ \hline \mathbf{K}_{21}^{1} & \mathbf{K}_{22}^{1} + \mathbf{K}_{22}^{2} & \mathbf{K}_{23}^{2} & & \\ \hline & \mathbf{K}_{32}^{2} & \mathbf{K}_{33}^{2} + \mathbf{K}_{33}^{3} & \mathbf{K}_{34}^{3} & \\ \hline & & \mathbf{K}_{43}^{3} & \mathbf{K}_{44}^{3} + \mathbf{K'}_{44}^{4} & \mathbf{K'}_{45}^{4} \\ \hline & & & \mathbf{K'}_{54}^{4} & \mathbf{K'}_{55}^{4} \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

which takes a new form of the stiffness matrix **K** after substituting their components for $\mathbf{K}_{ij}^{e}$. The new matrix **K** is presented on the next page.

Boundary conditions which we described when modifing the load vector lead to the matrix $\mathbf{K}^r$ presented on the successive page where all modified values are written in italic fonts.

The global stiffness matrix:

K=

	1			2			3			4			5		
118.7	0	-237.3	-118.6	0	-237.3	0	0	0	0	0	0	0	0	0	
0	84375.0	0	0	-84375.0	0	0	0	0	0	0	0	0	0	0	1
-237.3	0	632.8	237.3	0	316.4	0	0	0	0	0	0	0	0	0	
-118.7	0	237.3	64161.9	21324.0	166.2	-64043.2	-21324.0	-71.2	0	0	0	0	0	0	
0	-84375.0	0	21324.0	91554.2	213.5	-21324.0	-7179.2	213.5	0	0	0	0	0	0	2
-237.3	0	316.4	166.2	213.4	1581.5	71.2	-213.5	474.3	0	0	0	0	0	0	
0	0	0	-64043.2	-21324.0	71.2	107261.0	-7455.6	167.2	-43217.4	28779.6	96.0	0	0	0	
0	0	0	-21324.0	-7179.2	-213.5	-7455.6	26413.6	-69.4	28779.6	-19234.4	144.0	0	0	0	3
0	0	0	-71.2	213.4	474.3	167.2	-69.4	1780.7	-96.0	-144.0	416.0	0	0	0	
0	0	0	0	0	0	-43217.4	28779.6	-96.0	44166.6	-28779.6	853.2	-934.8	164.8	949.2	
0	0	0	0	0	0	28779.6	-19234.4	-144.0	-28779.6	187984.0	-144.0	-29303.1	-166186.0	0	4
0	0	0	0	0	0	96.0	144.0	416.0	853.2	-144.0	2097.7	-934.8	164.8	632.8	
0	0	0	0	0	0	0	0	0	-934.8	-29303.1	-934.8	6009.0	28695.6	-934.8	
0	0	0	0	0	0	0	0	0	164.8	-166186.0	164.8	28695.6	163690.0	164.8	5
0	0	0	0	0	0	0	0	0	949.2	0	632.8	-934.8	164.8	1265.6	

The global stiffness matrix after taking into consideration boundary conditions:

$$
\mathbf{K}^r = \begin{array}{cc}
\begin{array}{ccccc} \quad\quad 1 \quad\quad & \quad\quad\quad 2 \quad\quad\quad & \quad\quad\quad 3 \quad\quad\quad & \quad\quad\quad 4 \quad\quad\quad & \quad\quad 5 \quad\quad \end{array} & \\
\left[\begin{array}{ccc|ccc|ccc|ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 632.8 & 237.3 & 0 & 316.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 237.3 & 64161.9 & 21324.0 & 166.2 & -64043.2 & -21324.0 & -71.2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 21324.0 & 91554.2 & 213.5 & -21324.0 & -7179.2 & 213.5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 316.4 & 166.2 & 213.4 & 1581.5 & 71.2 & -213.5 & 474.3 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -64043.2 & -21324.0 & 71.2 & 107261.0 & -7455.6 & 167.2 & -43217.4 & 28779.6 & 96.0 & 0 & 0 & 0 \\
0 & 0 & 0 & -21324.0 & -7179.2 & -213.5 & -7455.6 & 26413.6 & -69.4 & 28779.6 & -19234.4 & 144.0 & 0 & 0 & 0 \\
0 & 0 & 0 & -71.2 & 213.4 & 474.3 & 167.2 & -69.4 & 1780.7 & -96.0 & -144.0 & 416.0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -43217.4 & 28779.6 & -96.0 & 44166.6 & -28779.6 & 853.2 & -934.8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 28779.6 & -19234.4 & -144.0 & -28779.6 & 187984.0 & -144.0 & -29303.1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 96.0 & 144.0 & 416.0 & 853.2 & -144.0 & 2097.7 & -934.8 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -934.8 & -29303.1 & -934.8 & 6009.0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
&
\begin{array}{c} \\ 1 \\ \\ \\ 2 \\ \\ \\ 3 \\ \\ \\ 4 \\ \\ \\ 5 \\ \\ \end{array}
\end{array}
$$

The set of equations of the finite element method:

$$\mathbf{K}^r\,\mathbf{u} = \mathbf{p}^r,$$

results in values of nodal displacements **u** and constraint reactions **r**:

$$\mathbf{u}=\left[\begin{array}{c} 0.0 \\ 0.0 \\ -0.11041E\text{-}1 \\ \hline 0.46553E\text{-}1 \\ 0.95211E\text{-}5 \\ -0.12832E\text{-}1 \\ \hline 0.52865E\text{-}1 \\ -0.16923E\text{-}1 \\ 0.35348E\text{-}2 \\ \hline 0.73664E\text{-}1 \\ 0.13642E\text{-}1 \\ 0.33103E\text{-}2 \\ \hline 0.7801E\text{-}1 \\ 0.0 \\ 0.0 \end{array}\right]\begin{array}{c} \\ 1 \\ \\ \\ 2 \\ \\ \\ 3 \\ \\ \\ 4 \\ \\ \\ 5 \\ \\ \end{array} \qquad \mathbf{r}=\left[\begin{array}{c} 0.142 \\ -0.803 \\ \text{-----} \\ \hline \text{-----} \\ \text{-----} \\ \text{-----} \\ \hline \text{-----} \\ \text{-----} \\ \text{-----} \\ \hline \text{-----} \\ \text{-----} \\ \text{-----} \\ \hline \text{-----} \\ 0.816 \\ 0.360 \end{array}\right]\begin{array}{c} \\ 1 \\ \\ \\ 2 \\ \\ \\ 3 \\ \\ \\ 4 \\ \\ \\ 5 \\ \\ \end{array}.$$

Nodal displacements after being transformed to local systems allow to determine forces in cross sections at nodes according to equation (4.9). These forces enable to calculate forces in any cross section within an element. Tab.4.E2.3 gives values of internal forces in frame elements determined in the cross sections which are distant from each other by 1/4 of the length of an element.

Fig.4.E2.2, Fig.4.E2.3, Fig.4.E2.4 and Fig.4.E2.5 present the graphs of displacements, normal and shearing forces and bending moments which have been drawn on the basis of the obtained results.

Tab.4.E2.3

No e	*x/L* [/]	*N* [kN]	*T* [kN]	*M* [kNm]
1	0.00	0.803	-0.142	0.0
	0.25	0.803	-0.142	-0.142
	0.50	0.803	-0.142	-0.283
	0.75	0.803	-0.142	-0.425
	1.00	0.803	-0.142	-0.567
2	0.00	0.120	-0.807	9.433
	0.25	0.120	-0.807	8.157
	0.50	0.120	-0.807	6.881
	0.75	0.120	-0.807	5.606
	1.00	0.120	-0.807	4.330
3	0.00	-0.564	-0.590	-0.077
	0.25	-0.564	-0.590	-1.140
	0.50	-0.564	-0.590	-2.204
	0.75	-0.564	-0.590	-3.267
	1.00	-0.564	-0.590	-4.330
4	0.00	-0.803	0.142	-0.361
	0.25	-0.803	0.142	-0.290
	0.50	-0.803	0.142	-0.219
	0.75	-0.803	0.142	-0.148
	1.00	-0.803	0.142	-0.077

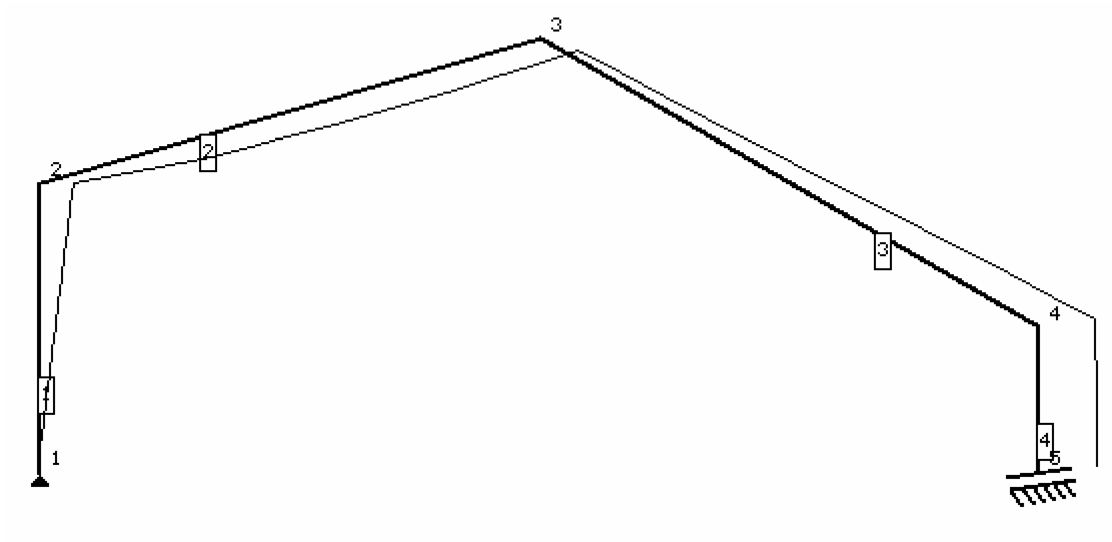

Fig.4.E2.2. The scheme of the deformed frame.

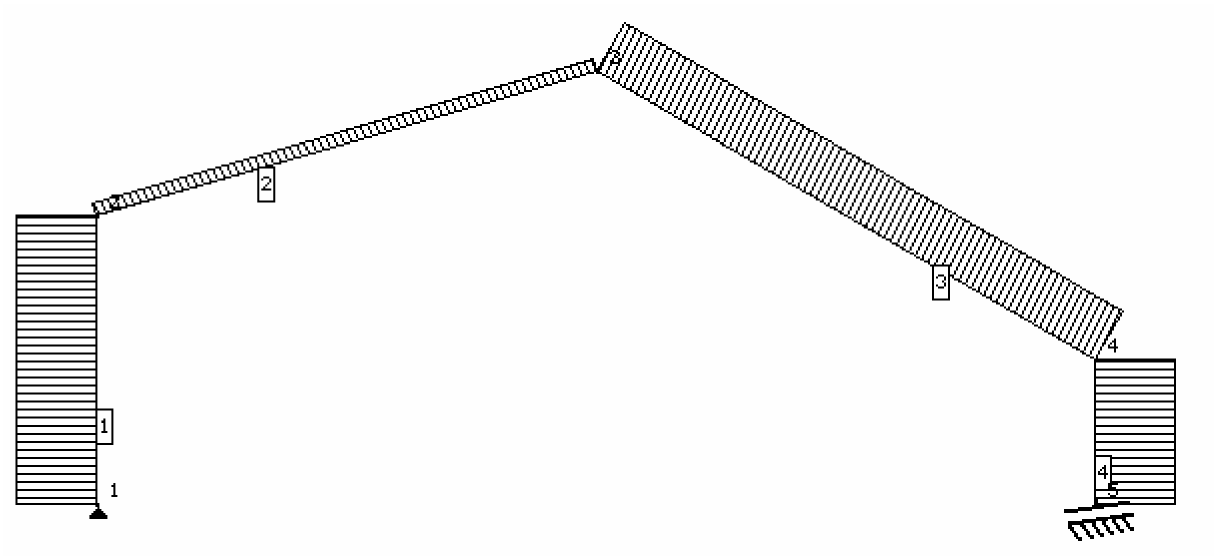

Fig.4.E2.3. The graph of normal forces.

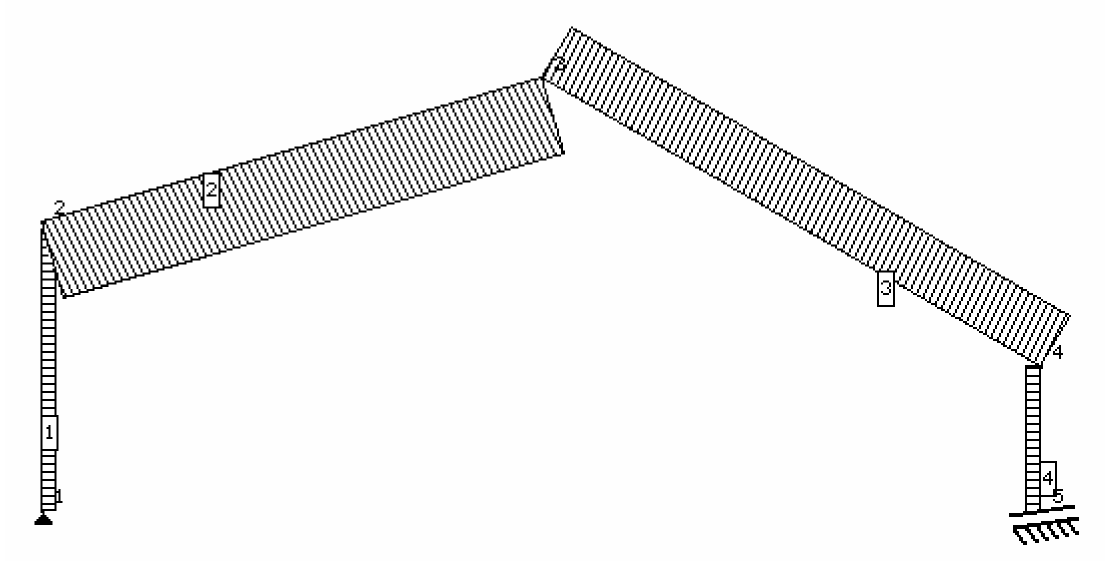

Fig.4.E2.4. The graph of shearing forces.

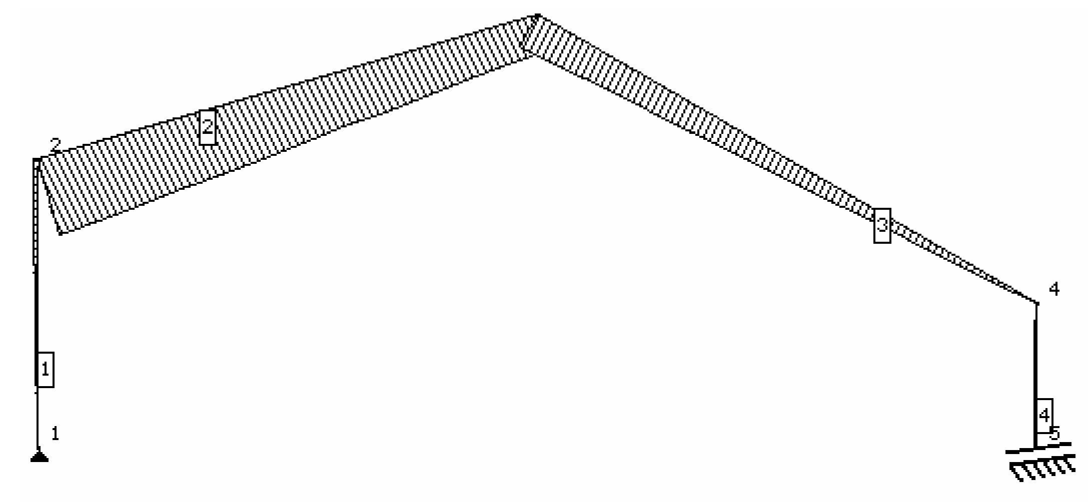

Fig.4.E2.5. The graph of bending moments.