CHAPTER VI.

STATICS OF TWO-DIMENSIONAL ELEMENTS

Previously described structures and elements used for modelling of these structures have contributed nothing new to the calculation method of the static bar structures apart from some ordering. The finite element method is only an adapted variant of the displacement method. It is so because of the simplicity of frame and truss structures. Differential equilibrium equations of frame and truss elements (4.16) are simple enough to be easily integrated. The situation in surface structures is completely different. Partial differential equations describing equilibrium of these structures have unique solutions for very simple problems only. Solutions obtained by an approximation method (for example, by expansion in a series) are very laborious and they require a lot of work to do and nevertheless a computer has to be used in order to solve a set of equations and sum series. In such a situation a numerical method which assumes some simplification at the stage of formation of element equilibrium equations appears to be more effective. That is why the finite element method has brought so many significant results to continuum mechanics. It can be easily noticed on the example of the simplest continuum which is a two-dimensional element. The 2D element can be defined as a solid of which one dimension (thickness) is considerably smaller than two other ones and its middle surface (the surface parallel to both external surfaces of an element) is a plane. A plate element has also such a shape has but a 2D element differs from a plate in the way a load acts in the central plate (Fig.6.1).

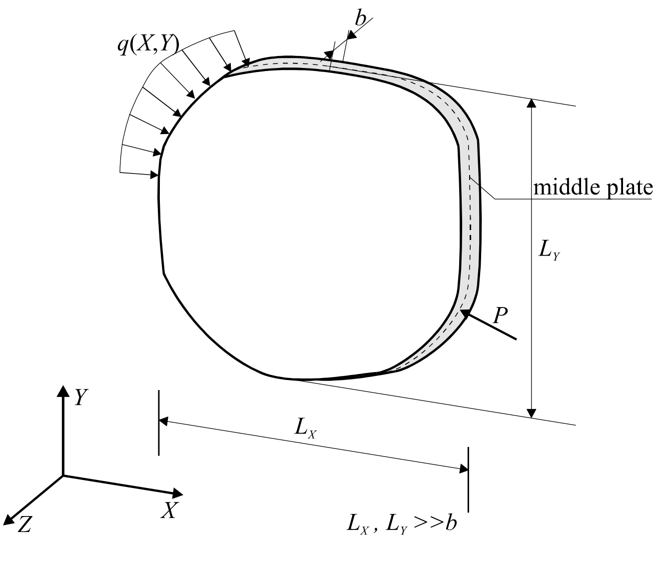

Fig.6.1

6.1. PLANE STRESS AND STRAIN

When external surfaces of a 2D element are free and this element is thin enough we can assume that $\sigma_z = 0, \tau_{zx} = 0, \tau_{zy} = 0$ in the whole thickness of the element. Then it is said that there is a plane stress in such a structure. The thinner the 2D element, the better approximation is (comp. [11], [17]). Hence only components of stress shown in Fig.6.2 can differ from zero.

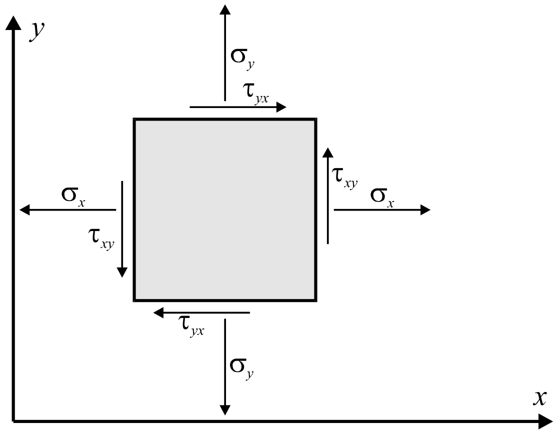

Fig.6.2

With regard to the symmetry of a stress tensor components of shear stress τ_{xy} and τ_{yx} are equal to each other, thus we have three independent components of stress which we compose in the stress vector:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}. \tag{6.1}$$

A completely different case of a structure with large thickness (Fig.6.1) can also be analysed by the method of a plane strain. Since the crosswise dimension of the structure shown in Fig.6.1 prevents from the deformation in the direction perpendicular to its cross section, then a thin layer cut out from this structure is in the state described by the equation:

$$\varepsilon_z = 0, \gamma_{zx} = 0, \gamma_{zy} = 0. \tag{6.2}$$

$\sigma_z \neq 0$ comes from the above equations, but the first equation allows to calculate the component σ_z on the basis of two other components of a direct stress. Thus, we have

$$\sigma_z = \nu(\sigma_x + \sigma_y), \tag{6.3}$$

which allows to limit the number of searched components of the stress vector to three components given in equation (6.1).

We also group independent components of the strain tensor in a column matrix which we have called a strain vector:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}. \tag{6.4}$$

There is a relation between vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ described by constitutive equations whose form depends on the model of a material which the structure is made of. In this book we deal only with elastic isotropic materials which apply to Hook's law. Hence we can write the constitutive equation as follows:

$$\boldsymbol{\sigma} = \mathbf{D}\,\boldsymbol{\varepsilon}, \tag{6.5}$$

where **D** is a quadratic matrix containing material elastic constants described in Chapter I.

For a plane stress the matrix **D** has the form written by equation (1.13). A plane strain requires another matrix for elastic constants which is described by (1.17).

6.2. GEOMETRIC RELATIONSHIPS

A certain point can move only on the plane during the deformation process and then the displacement vector of this point $\mathbf{u}(x,y)$ has two components:

$$\mathbf{u}(x,y) = \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix}. \tag{6.6}$$

Some known relations exist [17] between the components of displacement and strain vectors:

$$\varepsilon_x = \frac{\partial u_x}{\partial x},\ \varepsilon_y = \frac{\partial u_y}{\partial y},\ \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \tag{6.7}$$

which can be presented in the form:

$$\boldsymbol{\varepsilon} = \mathbf{D}\ \mathbf{u}(x,y), \tag{6.8}$$

where **D** is the matrix of differential operators (1.35).

6.3. THE STIFFNESS MATRIX OF AN ELASTIC ELEMENT

Let us divide a continuum into finite elements. We will discuss only a triangular 2D element in this book and we will choose such elements during discretization (Fig.6.3).

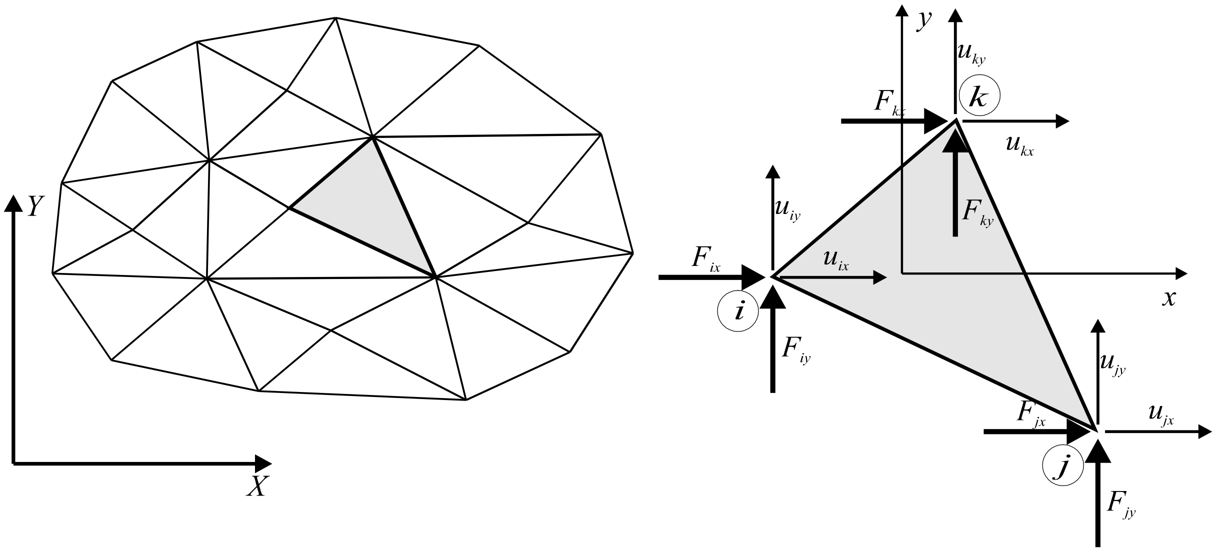

Fig.6.3

According to assumption (6.6) it is seen that every node of an element has two degrees of freedom and all nodal forces have two components, too. The local coordinate system xy is chosen in such way that its axes are parallel to the axes of the global coordinate system. Hence distinguishing components of local and global vectors and matrices is insignificant.

Now we group nodal displacements and forces in the vectors of:

- nodal and element displacements

$$\mathbf{u}_i = \begin{bmatrix} u_{ix} \\ u_{iy} \end{bmatrix}, \; \mathbf{u}_j = \begin{bmatrix} u_{jx} \\ u_{jy} \end{bmatrix}, \; \mathbf{u}_k = \begin{bmatrix} u_{kx} \\ u_{ky} \end{bmatrix}, \; \mathbf{u}^e = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \\ \mathbf{u}_k \end{bmatrix} = \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{jx} \\ u_{jy} \\ u_{kx} \\ u_{ky} \end{bmatrix} \tag{6.9}$$

- nodal and element forces

$$\mathbf{f}_i = \begin{bmatrix} F_{ix} \\ F_{iy} \end{bmatrix}, \; \mathbf{f}_j = \begin{bmatrix} F_{jx} \\ F_{jy} \end{bmatrix}, \; \mathbf{f}_k = \begin{bmatrix} F_{kx} \\ F_{ky} \end{bmatrix}, \; \mathbf{f}^e = \begin{bmatrix} \mathbf{f}_i \\ \mathbf{f}_j \\ \mathbf{f}_k \end{bmatrix} = \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{jx} \\ F_{jy} \\ F_{kx} \\ F_{ky} \end{bmatrix}. \tag{6.10}$$

Since we look for the dependence between nodal displacement and nodal forces vectors of an element we apply the principle of virtual work (comp. Chapter I) which requires giving the relation between displacements of points lying within the element and displacements of nodes. Accepting errors coming from approximation we assume that this relationship can be written by the function of two variables:

$$u_x(x,y) = N_i(x,y)u_{ix} + N_j(x,y)u_{jx} + N_k(x,y)u_{kx} \text{ and}$$
$$u_y(x,y) = N_i(x,y)u_{iy} + N_j(x,y)u_{jy} + N_k(x,y)u_{ky}\,, \tag{6.11}$$

or the tight matrix form:

$$\mathbf{u}(x,y) = \mathbf{N}^e(x,y)\,\mathbf{u}^e\,, \tag{6.12}$$

where $\mathbf{N}^e(x,y)$ is the matrix of shape functions of the element:

$$\mathbf{N}^e(x,y) = \left[N_i(x,y)\;\mathbf{I}\quad N_j(x,y)\;\mathbf{I}\quad N_k(x,y)\;\mathbf{I}\right], \tag{6.13}$$

and $N_i(x,y)$, $N_j(x,y)$, $N_k(x,y)$ are the shape functions for nodes i, j, k.

Let us now assume the simplest of all possible forms of the shape function for the node i

$$N_i(x,y) = a_i + b_i x + c_i y\,, \tag{6.14}$$

where a_i, b_i, c_i are constants which we determine on the basis of consistency conditions

$$N_i(x_i, y_i) = 1,\; N_i(x_j, y_j) = 0,\; N_i(x_k, y_k) = 0\,. \tag{6.15}$$

After inserting these conditions into equation (6.14), we obtain the set of equations:

$$\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{6.16}$$

which after being solved give the values of coefficients of the shape function.

Equation (6.16) can also be written in the general form:

$$\mathbf{M}\,\boldsymbol{\alpha}_i = \boldsymbol{\delta}_i\,, \text{ where } \boldsymbol{\delta}_i = \begin{bmatrix} \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \end{bmatrix} \tag{6.17}$$

which after modification depending on the change of i into j (or k) allows to determine the coefficients of the shape functions for the subsequent nodes. There is δ_{ij} in it which means Kronecker's delta in this equation.

We solve the set of equations (6.16) by the Cramer method

$$W = \det\mathbf{M} = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} = \begin{vmatrix} x_j & y_j \\ x_k & y_k \end{vmatrix} - \begin{vmatrix} x_i & y_i \\ x_k & y_k \end{vmatrix} + \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix},$$

$$W_{a_i} = \begin{vmatrix} 1 & x_i & y_i \\ 0 & x_j & y_j \\ 0 & x_k & y_k \end{vmatrix} = \begin{vmatrix} x_j & y_j \\ x_k & y_k \end{vmatrix}, \tag{6.18}$$

$$W_{b_i} = \begin{vmatrix} 1 & 1 & y_i \\ 1 & 0 & y_j \\ 1 & 0 & y_k \end{vmatrix} = -\begin{vmatrix} 1 & y_i \\ 1 & y_k \end{vmatrix} = y_j - y_k ,$$

$$W_{c_i} = \begin{vmatrix} 1 & x_i & 1 \\ 1 & x_j & 0 \\ 1 & x_k & 0 \end{vmatrix} = \begin{vmatrix} 1 & y_j \\ 1 & y_k \end{vmatrix} = x_k - x_j$$

then $a_i = \dfrac{W_{a_i}}{W}$, $b_i = \dfrac{W_{b_i}}{W}$, $c_i = \dfrac{W_{c_i}}{W}$.

Similarly, if we change the index i into j and we find $\delta_j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$,

$$W_{a_j} = \begin{vmatrix} 0 & x_i & y_i \\ 1 & x_j & y_j \\ 0 & x_k & y_k \end{vmatrix} = -\begin{vmatrix} x_i & y_i \\ x_k & y_k \end{vmatrix},$$

$$W_{b_j} = \begin{vmatrix} 1 & 0 & y_i \\ 1 & 1 & y_j \\ 1 & 0 & y_k \end{vmatrix} = y_k - y_i ,$$

$$W_{c_j} = \begin{vmatrix} 1 & x_i & 0 \\ 1 & x_j & 1 \\ 1 & x_k & 0 \end{vmatrix} = x_i - x_k , \tag{6.19}$$

$$a_j = \frac{W_{a_j}}{W},\ b_j = \frac{W_{b_j}}{W},\ c_j = \frac{W_{c_j}}{W}.$$

Finally, we have

$$\delta_k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$W_{a_k} = \begin{vmatrix} 0 & x_i & y_i \\ 0 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} = \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix},$$

$$W_{b_k} = \begin{vmatrix} 1 & 0 & y_i \\ 1 & 0 & y_j \\ 1 & 1 & y_k \end{vmatrix} = y_i - y_j \,, \qquad (6.20)$$

$$W_{c_k} = \begin{vmatrix} 1 & x_i & 0 \\ 1 & x_j & 0 \\ 1 & x_k & 1 \end{vmatrix} = x_j - x_i \,,$$

$$a_k = \frac{W_{a_k}}{W} \,,\; b_k = \frac{W_{b_k}}{W} \,,\; c_k = \frac{W_{c_k}}{W} \,.$$

for the node k.

As it is turns out constants a_i, a_j, a_k are insignificant for further transformations (because they are connected with the rigid motion of a 2D element) and they can be neglected when solving the set of equations (6.17).

After determining the shape functions of the element, let us come back to its deformation. We insert equation (6.12) in (6.8):

$$\boldsymbol{\varepsilon} = \mathbf{D}\ \mathbf{N}^e(x,y)\,\mathbf{u}^e = \mathbf{B}^e(x,y)\,\mathbf{u}^e \,, \qquad (6.21)$$

obtaining the dependence between the nodal displacements of the element and its strains. The matrix $\mathbf{B}$ existing in equation (6.21) is called a geometric matrix and it can be expressed as follows:

$$\mathbf{B}^e(x,y) = \begin{bmatrix} \mathbf{B}_i(x,y) & \mathbf{B}_j(x,y) & \mathbf{B}_k(x,y) \end{bmatrix},$$

where $\mathbf{B}_n = \mathbf{D}\ \mathbf{N}_n(x,y) = \begin{bmatrix} b_n & 0 \\ 0 & c_n \\ c_n & b_n \end{bmatrix}$ (6.22)

is the geometric matrix of any node n.

Thus, we have all components which are necessary to write an element equilibrium equation. We apply the principle of virtual work which says that the external work (done by external forces - here nodal forces) has to be equal to internal work (done by stress) of a 2D element:

$$\left(\mathbf{u}^e\right)^{\mathrm{T}} \mathbf{f}^e = \int_V \boldsymbol{\varepsilon}^{\mathrm{T}} \boldsymbol{\sigma}\, dV \,. \qquad (6.23)$$

We transform this equation first substituting the constitutive relation (6.5) for $\boldsymbol{\sigma}$ and next substituting geometric relations (6.21) for $\boldsymbol{\varepsilon}$:

$$\left(\mathbf{u}^e\right)^{\mathrm{T}}\mathbf{f}^e = \int_V \left(\mathbf{B}^e\mathbf{u}^e\right)^{\mathrm{T}}\mathbf{D}\mathbf{B}^e\mathbf{u}^e dV = \left(\mathbf{u}^e\right)^{\mathrm{T}}\int_V \left(\mathbf{B}^e\right)^{\mathrm{T}}\mathbf{D}\mathbf{B}^e dV\,\mathbf{u}^e . \tag{6.24}$$

In this equation the nodal displacement vectors of the element as being independent of variables x and y are taken to the front and back of the integral. Equation (6.24) can be realised independently of element displacements only when

$$\mathbf{f}^e = \int_V \left(\mathbf{B}^e\right)^{\mathrm{T}}\mathbf{D}\,\mathbf{B}^e dV\,\mathbf{u}^e , \tag{6.25}$$

which after the comparison with the known relation was referred to in all previous chapters of this book:

$$\mathbf{f}^e = \mathbf{K}^e\,\mathbf{u}^e ,$$

gives us the equation determining coefficients of the element stiffness matrix:

$$\mathbf{K}^e = \int_V \left(\mathbf{B}^e\right)^{\mathrm{T}}\mathbf{D}\,\mathbf{B}^e dV . \tag{6.26}$$

Building the element stiffness matrix can be considerably simple if we note that this matrix divides into blocks:

$$\mathbf{K}^e = \begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{ij} & \mathbf{K}_{ik} \\ \mathbf{K}_{ji} & \mathbf{K}_{jj} & \mathbf{K}_{jk} \\ \mathbf{K}_{ki} & \mathbf{K}_{kj} & \mathbf{K}_{kk} \end{bmatrix}, \tag{6.27}$$

in which any of them, for example $\mathbf{K}_{ij}$, can be calculated from the equation:

$$\mathbf{K}_{ij} = \int_V \left(\mathbf{B}_i\right)^{\mathrm{T}}\mathbf{D}\,\mathbf{B}_j dV , \tag{6.28}$$

and others coming from analogous equations formed after suitable changes of indices have been made.

The insertion of the geometric matrices $\mathbf{B}_i$ and $\mathbf{B}_j$ given by equation (6.22) and the matrix $\mathbf{D}$ given by equation (1.13) into (6.28) results in

$$\mathbf{K}_{ij} = \left(\mathbf{B}_i\right)^{\mathrm{T}}\mathbf{D}\,\mathbf{B}_j \int_V dV = \left(\mathbf{B}_i\right)^{\mathrm{T}}\mathbf{D}\,\mathbf{B}_j Ab =$$

$$= \frac{EAb}{1-\nu^2}\begin{bmatrix} b_i b_j + c_i c_j \dfrac{1-\nu}{2} & b_i c_j \nu + b_j c_i \dfrac{1-\nu}{2} \\ b_j c_i \nu + b_i c_j \dfrac{1-\nu}{2} & c_i c_j + b_i b_j \dfrac{1-\nu}{2} \end{bmatrix}, \tag{6.29}$$

where A is the surface of a 2D element; b is the thickness of 2D element.

The above matrix is the block of the stiffness matrix for plane stress.

Let us note that matrices $\mathbf{B}_i$, $\mathbf{B}_j$ and $\mathbf{D}$ do not contain components dependent on variables x, y, z, thus we can take them to the front of the sign of the integral.

We obtain the block of the stiffness matrix for plane strain accepting the matrix of material constants according to (1.17):

$$\mathbf{K}_{ij} = \frac{EAb}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) b_i b_j + c_i c_j \dfrac{1-2\nu}{2} & b_i c_j \nu + b_j c_i \dfrac{1-2\nu}{2} \\ b_j c_i \nu + b_i c_j \dfrac{1-2\nu}{2} & (1-\nu) c_i c_j + b_i b_j \dfrac{1-2\nu}{2} \end{bmatrix}. \tag{6.30}$$

Since the local coordinate system is assumed in such a way that its axes are parallel to the global coordinate system, then we do not have to transform the obtained stiffness matrix.

6.4. ELEMENT STRAIN AND STREES

We also calculate element strains. They are given by equation (6.21) and taking into consideration equation (6.22) we have

$$\varepsilon_x = \sum_{n=i,j,k} b_n u_{nx} \,,\; \varepsilon_y = \sum_{n=i,j,k} b_n u_{ny} \,,\; \gamma_{xy} = \sum_{n=i,j,k} \left(c_n u_{nx} + b_n u_{ny} \right). \tag{6.31}$$

We see that components of the strain vector are constant within the element which is the consequence of the assumption of linear shape functions. This element is called CST (*constant strain triangle*).

We determine element stresses from constitutive equation (6.5) and equation (1.13) or (1.17) according to the kind of variant that we deal with. It is obvious that strains just as stresses are constant within the CST element.

6.5. A NODAL FORCE VECTOR FROM A DISTRIBUTED LOAD

Loads of 2D elements can be treated as loads of plane trusses which means that they can be applied to the nodes of a structure. But if a distributed load acting on the boundary of an element is given, then it should be led to concentrated forces acting on the nodes of an element (Fig.6.4).

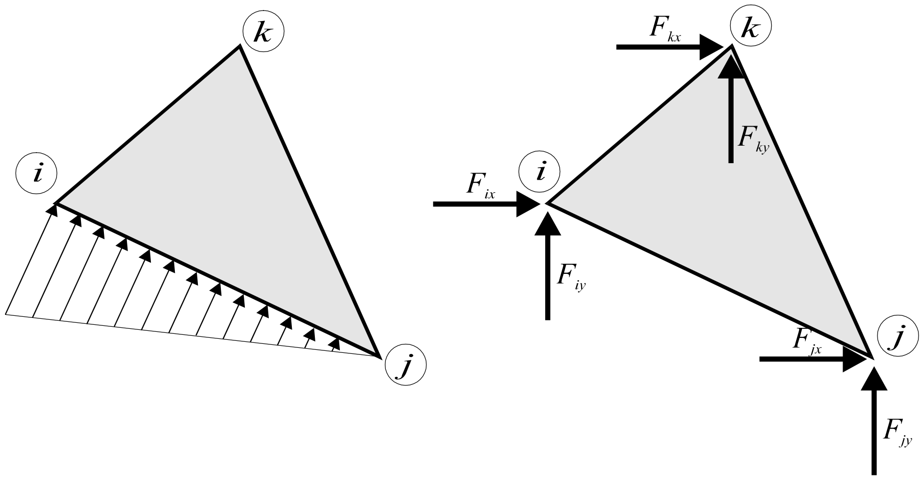

Fig.6.4

Similarly, as in previous chapters we apply the principal of virtual work giving the following equilibrium equation for this case:

$$\left(\mathbf{u}^e\right)^{\mathrm{T}}\mathbf{f}^e + L_{ij}\int_0^1 \mathbf{u}(\xi)^{\mathrm{T}}\mathbf{q}(\xi)d\xi = 0\,, \tag{6.32}$$

where $\mathbf{u}(\xi)$ contains functions describing the displacement of the loaded edge and $\mathbf{q}(\xi) = \begin{bmatrix} q_x(\xi) \\ q_y(\xi) \end{bmatrix}$ contains functions describing the load on the edge, L_{ij} is the length of the edge i-j, ξ is the nondimensional coordinate taking value zero at the node i and value 1 at the node j. Since we assume linear shape functions for the element, then we write the vector $\mathbf{u}(\xi)$ as follows:

$$\mathbf{u}(\xi) = \mathbf{N}^e_{ij}\mathbf{u}^e\,, \tag{6.33}$$

where $\mathbf{N}^e_{ij}$ is the matrix of shape functions for displacements of the border .

$$\mathbf{N}^e_{ij} = \left[N^o_i(\xi)\,\mathbf{I} \quad N^o_j(\xi)\,\mathbf{I} \quad N^o_k(\xi)\,\mathbf{0}\right], \tag{6.34}$$

where $N^o_i(\xi) = 1-\xi$, $N^o_j(\xi) = \xi$,

or in the developed form

$$\mathbf{N}^e_{ij} = \begin{bmatrix} 1-\xi & 0 & \xi & 0 & 0 & 0 \\ 0 & 1-\xi & 0 & \xi & 0 & 0 \end{bmatrix}. \tag{6.35}$$

After inserting relation (6.33) into equation (6.32), we obtain

$$\mathbf{f}^e = -L_{ij}\int_0^1 \left(\mathbf{N}_{ij}^e\right)^{\mathrm{T}} \mathbf{q}(\xi)d\xi, \tag{6.36}$$

After taking into consideration the shape functions described by equation (6.35), we obtain

$$\mathbf{f}^e = -L_{ij}\int_0^1 \begin{bmatrix} (1-\xi)q_x(\xi) \\ (1-\xi)q_y(\xi) \\ \xi q_x(\xi) \\ \xi q_y(\xi) \\ 0 \\ 0 \end{bmatrix} d\xi. \tag{6.37}$$

For example, let us calculate, the nodal force vector due to the linear distributed load on the edge *i-j* of value q_{ix}, q_{iy} - at the node *i* and q_{jx}, q_{jy} - at the node *j*. We write such a load with the help of a nondimensional coordinate ξ:

$$\mathbf{q}(\xi) = \begin{bmatrix} q_{ix}(1-\xi) + q_{jx}\xi \\ q_{iy}(1-\xi) + q_{jy}\xi \end{bmatrix}, \tag{6.38}$$

and after inserting the above equation into equation (6.37), we obtain

$$\mathbf{f}^e = -L_{ij} \begin{bmatrix} q_{ix}\int_0^1 (1-\xi)^2 d\xi + q_{jx}\int_0^1 (1-\xi)\xi d\xi \\ q_{iy}\int_0^1 (1-\xi)^2 d\xi + q_{jy}\int_0^1 (1-\xi)\xi d\xi \\ q_{ix}\int_0^1 (1-\xi)\xi d\xi + q_{jx}\int_0^1 \xi^2 d\xi \\ q_{iy}\int_0^1 (1-\xi)\xi d\xi + q_{jy}\int_0^1 \xi^2 d\xi \\ 0 \\ 0 \end{bmatrix}, \tag{6.39}$$

which after integration gives

$$\mathbf{f}^e = -\frac{L_{ij}}{6} \begin{bmatrix} 2q_{ix} + q_{jx} \\ 2q_{iy} + q_{jy} \\ q_{ix} + 2q_{jx} \\ q_{iy} + 2q_{jy} \\ 0 \\ 0 \end{bmatrix}. \tag{6.40}$$

For a particular case when the load is constant and equal to $\mathbf{q}(\xi)=\begin{bmatrix} q_{ox} \\ q_{oy} \end{bmatrix}$, on the basis of equation (6.40) we obtain

$$\mathbf{f}^{e} = -\frac{L_{ij}}{2}\begin{bmatrix} q_{ox} \\ q_{oy} \\ q_{ox} \\ q_{oy} \\ 0 \\ 0 \end{bmatrix}. \tag{6.41}$$

It should be remembered that the calculated forces are forces acting on the element. We obtain the necessary nodal forces changing the senses of vectors which means:

$$\mathbf{p}^{e} = -\mathbf{f}^{e}, \tag{6.42}$$

where $\mathbf{p}^{e}$ is the nodal force vector for the nodes touching the element *e*.

6.6. A NODAL FORCE VECTOR DUE TO A TEMPERATURE LOAD

As in the previous section, we apply the principal of virtual work to calculate alternative nodal forces replacing a temperature load. In accordance with the features of a CST element we will take into consideration only a constant temperature field within the element.

The suitable equation of virtual work has the form:

$$\left(\mathbf{u}^{e}\right)^{\mathrm{T}}\mathbf{f}^{et} = \int_{V}\varepsilon^{\mathrm{T}}\sigma_{t}\,dV = \int_{V}\varepsilon^{\mathrm{T}}\mathbf{D}\,\varepsilon_{t}\,dV, \tag{6.43}$$

where $\boldsymbol{\sigma}_t$ is the stress field in the element which is caused by the temperature and $\boldsymbol{\varepsilon}_t$ is the strain of the element caused by the change of a temperature.

Assuming isotropy of a 2D element we obtain

$$\boldsymbol{\varepsilon}_t = \alpha_t\Delta t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \tag{6.44}$$

After inserting geometric relation (6.21) into equation (6.43), we obtain

$$\mathbf{f}^{et} = \alpha_t\Delta t\int_{V}\left(\mathbf{B}^{e}\right)^{\mathrm{T}}\mathbf{D}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}dV = \alpha_t\Delta t A b\left(\mathbf{B}^{e}\right)^{\mathrm{T}}\mathbf{D}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \tag{6.45}$$

For the plane stress this equation is simplified to the following relation:

$$\mathbf{f}_{\mathrm{PSN}}^{et} == \frac{\alpha_t \Delta t E A b}{1-\nu}\begin{bmatrix} b_i \\ c_i \\ b_j \\ c_j \\ b_k \\ c_k \end{bmatrix}, \tag{6.46}$$

where b_i ... c_k are coefficients of shape functions of the CST element.

Plane strain gives a bit different nodal force vector:

$$\mathbf{f}_{\mathrm{PSO}}^{et} == \frac{\alpha_t \Delta t E A b}{(1+\nu)(1-2\nu)}\begin{bmatrix} b_i \\ c_i \\ b_j \\ c_j \\ b_k \\ c_k \end{bmatrix}. \tag{6.47}$$

As in previous sections, we should change the signs of components of nodal forces before applying them to the nodes:

$$\mathbf{p}^{et} = -\mathbf{f}^{et}. \tag{6.48}$$

We calculate stresses in the element undergoing the action of a temperature taking into consideration strains caused by the thermal expansion of the element:

$$\boldsymbol{\sigma}_t = \mathbf{D}(\varepsilon - \varepsilon_t) = \mathbf{D}\left(\mathbf{B}\,\mathbf{u}^e - \alpha_t \Delta t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right). \tag{6.49}$$

6.7. BOUNDARY CONDITIONS OF A 2D ELEMENT

Boundary conditions of a two-dimensional structure can be treated analogously to the conditions in a plane truss because the nodes of both systems have two degrees of freedom on the plane XY.

Hence we have: fixed supports (at the node r_1 Fig.6.5) and supports which can move along the X axis (at the node r_2), next supports which can move along the Y axis (at the node r_4) or skew supports (at the node r_3). The boundary conditions for these supports are as follows:

- node r_1: $u_{r_1X} = 0$, $u_{r_1Y} = 0$,
- node r_2: $u_{r_2Y} = 0$,

– node r_4: $u_{r_4 X} = 0$,

– for node r_3, where constraints are not in accord with the axes of the global coordinate system we propose the use of boundary elements described in Chapter II.

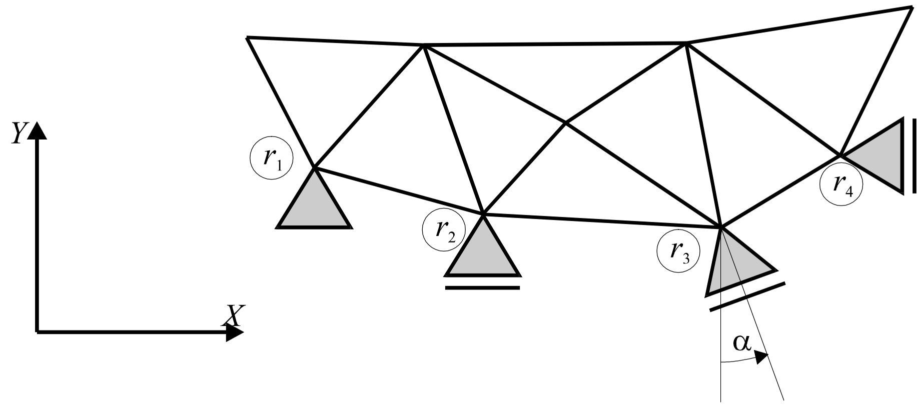

Fig.6.5