Example 4.E1.

The content of the example

A 2D frame structure with a static scheme shown in Fig.4.E1.1 is loaded with two concentrated forces and a distributed load. Elements No 1 and No 3 have quadratic cross sections with one side having the length of 15 cm, and a lock (element No 2) has a rectangular cross section with dimensions 15 x 20 cm. The material from which the elements are made is characterised by Young's modulus $E=1.5 \cdot 10^7$ kPa.

Determine nodal displacements (displacements u_{iX}, u_{iY} and rotations φ_i) and draw graphs of internal forces for the frame elements.

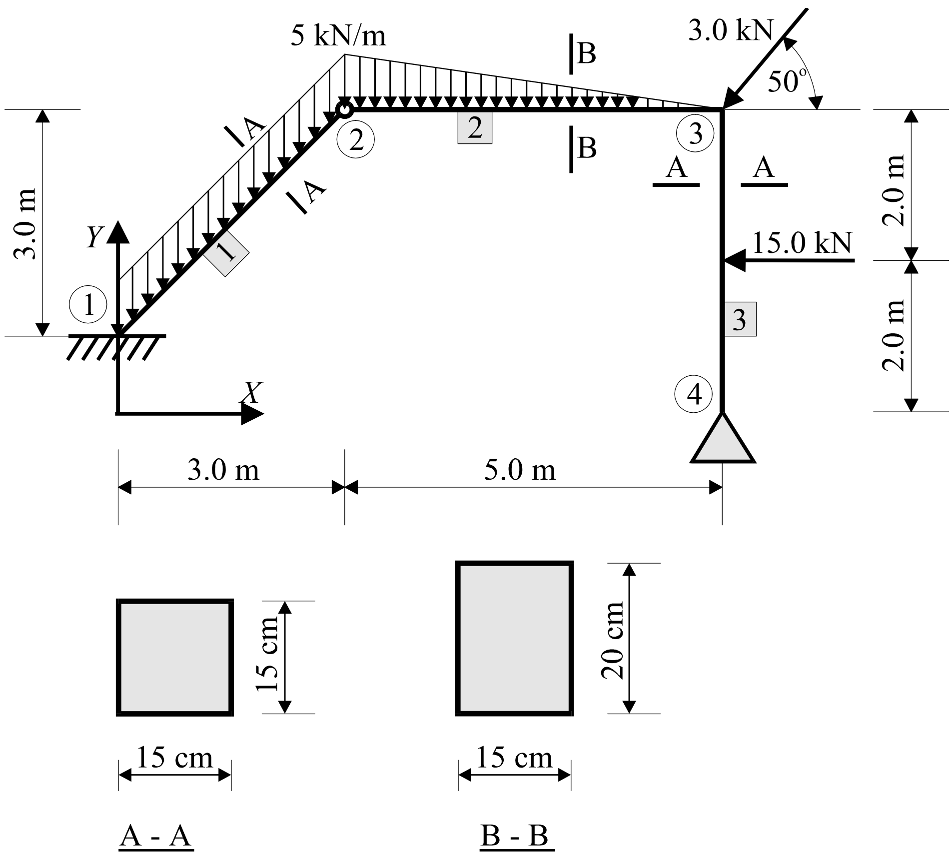

Fig.4.E1.1

The solution of the example

We start solving the problem from collecting necessary data. Nodal coordinates in the global system (XY - shown in Fig.4.E1.1) and nodal loads are contained in Tab.4.E1.1. Tab.4.E1.2 contains data related to structure elements:

- numbers of first and last nodes (n_i, n_j),
- lengths of element projections on axes X and Y (L_X, L_Y),
- lengths of elements (L_n),
- angles of inclination of elements with regard to the X axis,
- areas (A) of cross sections of elements,
- moments of inertia of cross sections with regard to the Z axis (J_z).

Tab.4.E1.1

Node No n	X_n [m]	Y_n [m]	P_{Xn} [kN]	P_{Yn} [kN]	M_n [kNm]
1	0.0	1.0	------	------	------
2	3.0	4.0	------	------	------
3	8.0	4.0	-1.9284	-2.2981	------
4	8.0	0.0	------	------	------

Tab.4.E1.2

No n	Node No n_i	n_j	L_{nX} [m]	L_{nY} [m]	L_n [m]	α_n [rad]	Cross section	A [m^2]	J_z [m^4]
1	1	2	3.0	3.0	4.2426	0.785398	A-A	$2.25 \cdot 10^{-2}$	$4218.75 \cdot 10^{-8}$
2	2	3	5.0	0.0	5.0	0.0	B-B	$3.0 \cdot 10^{-2}$	$1.0 \cdot 10^{-4}$
3	4	3	0.0	4.0	4.0	1.570796	A-A	$2.25 \cdot 10^{-2}$	$4218.75 \cdot 10^{-8}$

Finding vectors of nodal forces caused by external forces is the next stage. We use equation (4.53) to obtain components of the vectors $\mathbf{f}'^1$, $\mathbf{f}'^2$, $\mathbf{f}'^3$ in the local system, and next we transform them to the global system according to equation (2.31) using the element rotation matrix $\mathbf{R}^e$ coming from equation (4.26). After transformation we obtain the following vectors $\mathbf{f}^1$, $\mathbf{f}^2$, $\mathbf{f}^3$:

$$\mathbf{f}'^1 = \left[\begin{array}{c} 7.5 \\ 7.5 \\ 5.3032 \\ \hline 7.5 \\ 7.5 \\ -5.3032 \end{array}\right] \begin{array}{c} \\ 1 \\ \\ \\ 2 \\ \\ \end{array} \qquad \mathbf{f}^1 = \left[\begin{array}{c} 0.0 \\ 10.6065 \\ 5.3032 \\ \hline 0.0 \\ 10.6065 \\ -5.3032 \end{array}\right] \begin{array}{c} \\ 1 \\ \\ \\ 2 \\ \\ \end{array}$$

$$\mathbf{f}'^2 = \left[\begin{array}{c} 0.0 \\ 8.75 \\ 6.25 \\ \hline 0.0 \\ 3.75 \\ -4.1667 \end{array}\right] \begin{array}{c} \\ 2 \\ \\ \\ 3 \\ \\ \end{array} \qquad \mathbf{f}^2 = \left[\begin{array}{c} 0.0 \\ 8.75 \\ 6.25 \\ \hline 0.0 \\ 3.75 \\ -4.1667 \end{array}\right] \begin{array}{c} \\ 2 \\ \\ \\ 3 \\ \\ \end{array}$$

$$\mathbf{f}'^3 = \left[\begin{array}{c} 0.0 \\ -7.5 \\ 7.5 \\ \hline 0.0 \\ -7.5 \\ 7.5 \end{array}\right] \begin{array}{c} \\ 4 \\ \\ \\ 3 \\ \\ \end{array} \qquad \mathbf{f}^3 = \left[\begin{array}{c} 7.5 \\ 0.0 \\ -7.5 \\ \hline 7.5 \\ 0.0 \\ 7.5 \end{array}\right] \begin{array}{c} \\ 4 \\ \\ \\ 3 \\ \\ \end{array}$$

Equation (4.27) allows to determine element stiffness matrices in the global coordinate system. Hence we obtain

$$\begin{array}{cc} & \begin{array}{cccccc} & 1 & & & 2 & \end{array} \\ \mathbf{K}^1 = & \left[\begin{array}{ccc|ccc} 39824.5 & 39725.0 & -149.157 & -39824.5 & -39725.0 & -149.16 \\ 39725.0 & 39824.5 & 149.157 & -39725.0 & -39824.5 & 149.16 \\ -149.16 & 149.16 & 596.63 & 149.16 & -149.16 & 298.31 \\ \hline -39824.5 & -39725.0 & 149.167 & 39824.5 & 39725.0 & 149.16 \end{array}\right. \end{array} \begin{array}{c} \\ 1 \\ \\ \\ \end{array}$$

$$\begin{array}{cc}\left[\begin{array}{ccc|ccc} -39725.0 & -39824.5 & -149.16 & 39725.0 & 39824.5 & -149.16 \\ -149.16 & 149.157 & 298.31 & 149.16 & -149.16 & 596.63 \end{array}\right] & 2\end{array}$$

$$\begin{array}{ccc} & \begin{array}{cc} \qquad\qquad 2 \qquad\qquad\qquad & \qquad\qquad 3 \qquad\qquad \end{array} & \\ \mathbf{K}^2= & \left[\begin{array}{ccc|ccc} 90000.0 & 0.0 & 0.0 & -90000.0 & 0.0 & 0.0 \\ 0.0 & 144.0 & 360.0 & 0.0 & -144.0 & 360.0 \\ 0.0 & 360.0 & 1200.0 & 0.0 & -360.0 & 600.0 \\ \hline -90000.0 & 0.0 & 0.0 & 90000.0 & 0.0 & 0.0 \\ 0.0 & -144.0 & -360.0 & 0.0 & 144.0 & -360.0 \\ 0.0 & 360.0 & 600.0 & 0.0 & -360.0 & 1200.0 \end{array}\right] & \begin{array}{c} \\ 2 \\ \\ \\ 3 \\ \\ \end{array} \end{array}$$

$$\begin{array}{ccc} & \begin{array}{cc} \qquad\qquad 4 \qquad\qquad\qquad & \qquad\qquad 3 \qquad\qquad \end{array} & \\ \mathbf{K}^3= & \left[\begin{array}{ccc|ccc} 118.65 & 0.0 & -237.31 & -118.65 & 0.0 & -237.31 \\ 0.0 & 84375.0 & 0.0 & 0.0 & -84375.0 & 0.0 \\ -237.31 & 0.0 & 632.82 & 237.31 & 0.0 & 316.41 \\ \hline -118.65 & 0.0 & 237.31 & 118.65 & 0.0 & 237.31 \\ 0.0 & -84375.0 & 0.0 & 0.0 & 84375.0 & 0.0 \\ -237.31 & 0.0 & 316.41 & 237.31 & 0.0 & 632.82 \end{array}\right] & \begin{array}{c} \\ 4 \\ \\ \\ 3 \\ \\ \end{array} \end{array}$$

Since element No 1 is connected with the neighbouring element with the help of articulated joint, we do the static condensation of the stiffness matrix $\mathbf{K}^1$ by eliminating the third degree of freedom (the nodal rotation angle). We apply equation (4.41)

$$\mathbf{K}^{1\,(0,3)} = \mathbf{K}_{11}^{1} - \mathbf{K}_{10}^{1}\left(\mathbf{K}_{00}^{1}\right)^{-1}\left(\mathbf{K}_{10}^{1}\right)^{\mathrm{T}}$$

where

$$\mathbf{K}_{11}^{1} = \begin{bmatrix} 39824.5 & 39725.0 & -149.16 & -39824.5 & -39725.0 \\ 39725.0 & 39824.5 & 149.16 & -39725.0 & -39824.5 \\ -149.16 & 149.16 & 596.63 & 149.16 & -149.16 \\ -39824.5 & -39725.0 & 149.16 & 39824.5 & 39725.0 \\ -39725.0 & -39824.5 & -149.16 & 39725.0 & 39824.5 \end{bmatrix}$$

$$\mathbf{K}_{10}^{1} = \begin{bmatrix} -149.16 \\ 149.16 \\ 298.31 \end{bmatrix}$$

$$\begin{bmatrix} 149.16 \\ -149.16 \end{bmatrix}$$

$$\mathbf{K}_{01}^{1} = \begin{bmatrix} -149.16 & 149.16 & 298.31 & 149.16 & -149.16 \end{bmatrix}$$

$\mathbf{K}_{00}^{1} = 596.628$

Matrix multiplication leads to the following form:

$$\mathbf{K}^{1\,(0,3)} = \begin{array}{c c} \begin{array}{cccccc} & 1 & & & 2 & \end{array} & \\ \left[\begin{array}{ccc|ccc} 39787.2 & 39762.3 & -74.58 & -39787.2 & -39762.3 & 0 \\ 39762.3 & 39787.2 & 74.58 & -39762.3 & -39787.2 & 0 \\ -74.58 & 74.58 & 447.47 & 74.58 & -74.58 & 0 \\ \hline -39787.2 & -39762.3 & 74.58 & 39787.2 & 39762.3 & 0 \\ -39762.3 & -39787.2 & -74.58 & 39762.3 & 39787.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] & \begin{array}{c} \\ 1 \\ \\ \\ 2 \\ \\ \end{array} \end{array},$$

Since element No 1 is loaded, we have to do the static condensation of the nodal load vector $\mathbf{f}^{1}$, too. We apply equation (4.44) obtaining

$$\mathbf{f}^{1(0,3)} = \begin{array}{cc} \left[\begin{array}{c} -1.3258 \\ 11.9323 \\ 7.9549 \\ \hline 1.3258 \\ 9.2807 \\ 0.0 \end{array}\right] & \begin{array}{c} \\ 1 \\ \\ \\ 2 \\ \\ \end{array} \end{array}$$

After aggregation of the global vector of nodal forces we obtain

$$
\mathbf{p}=\left[\begin{array}{c} 0.0 \\ 0.0 \\ 0.0 \\ \hline 0.0 \\ 0.0 \\ 0.0 \\ \hline -1.9284 \\ -2.2981 \\ 0.0 \\ \hline 0.0 \\ 0.0 \\ 0.0 \end{array}\right]
-\left[\begin{array}{c} -1.3258 \\ 11.9323 \\ 7.9549 \\ \hline 1.3258 \\ 9.2807 \\ 0.0 \\ \hline \\ \\ \\ \hline \\ \\ \\ \end{array}\right]
-\left[\begin{array}{c} \\ \\ \\ \hline 0.0 \\ 8.75 \\ 6.25 \\ \hline 0.0 \\ 3.75 \\ -4.1667 \\ \hline \\ \\ \\ \end{array}\right]
-\left[\begin{array}{c} \\ \\ \\ \hline \\ \\ \\ \hline 7.5 \\ 0.0 \\ 7.5 \\ \hline 7.5 \\ 0.0 \\ -7.5 \end{array}\right]
=\left[\begin{array}{c} 1.3258 \\ -11.9323 \\ -7.9549 \\ \hline -1.3258 \\ -18.0307 \\ -6.25 \\ \hline -9.4284 \\ -6.0481 \\ -3.3333 \\ \hline -7.5 \\ 0.0 \\ 7.5 \end{array}\right]
\begin{array}{c} \\ 1 \\ \\ \\ 2 \\ \\ \\ 3 \\ \\ \\ 4 \\ \\ \end{array}
$$

Consideration of boundary conditions depends on replacing suitable components of the vector **p** (these components which correspond constraints from supports, that is 1, 2, 3, 10, 11) by zeros which results in the vector:

$$
\mathbf{p}^r=\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline -1.3258 \\ -18.0307 \\ -6.25 \\ \hline -9.4284 \\ -6.0481 \\ -3.3333 \\ \hline 0 \\ 0 \\ 7.5 \end{array}\right]
\begin{array}{c} \\ 1 \\ \\ \\ 2 \\ \\ \\ 3 \\ \\ \\ 4 \\ \\ \end{array}
$$

The aggregation of the stiffness matrix of the structure leads to the equation:

$$\mathbf{K}=\begin{array}{cc} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \left[\begin{array}{c|c|c|c} \mathbf{K}_{11}^{1(0,3)} & \mathbf{K}_{12}^{1(0,3)} & & \\ \hline \mathbf{K}_{21}^{1(0,3)} & \mathbf{K}_{22}^{1(0,3)}+\mathbf{K}_{22}^{2} & \mathbf{K}_{23}^{2} & \\ \hline & \mathbf{K}_{32}^{2} & \mathbf{K}_{33}^{2}+\mathbf{K}_{33}^{3} & \mathbf{K}_{34}^{3} \\ \hline & & \mathbf{K}_{43}^{3} & \mathbf{K}_{44}^{3} \end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

The final form of the global stiffness matrix is presented on the next page.

After taking into consideration the boundary conditions (inserting zeros into rows and columns numbered 1, 2, 3, 10 and 11 and inserting 1 into these rows on the main diagonal), we obtain the matrix $\mathbf{K}^r$ which is presented on the successive page after the matrix **K**.

Solving the set of equations

$$\mathbf{K}^r\,\mathbf{u}=\mathbf{p}^r$$

gives us nodal displacements of the frame contained in the vector **u** and next we calculate the constraint reaction vector **r** applying equation (2.75)

$$\mathbf{u}=\left[\begin{array}{c} 0.0 \\ 0.0 \\ 0.0 \\ \hline 0.21315\text{E-}1 \\ -0.21679\text{E-}1 \\ 0.22045\text{E-}2 \\ \hline 0.21174\text{E-}1 \\ -0.10712\text{E-}3 \\ -0.18825\text{E-}2 \\ \hline 0.0 \\ 0.0 \\ 0.48527\text{E-}2 \end{array}\right]\begin{array}{c} \\ 1 \\ \\ \\ 2 \\ \\ \\ 3 \\ \\ \\ 4 \\ \\ \end{array} \qquad \mathbf{r}=\left[\begin{array}{c} 12.646 \\ 26.973 \\ 11.162 \\ \hline \text{-----} \\ \text{-----} \\ \text{-----} \\ \hline \text{-----} \\ \text{-----} \\ \text{-----} \\ \hline 4.283 \\ 9.039 \\ \text{-----} \end{array}\right]\begin{array}{c} \\ 1 \\ \\ \\ 2 \\ \\ \\ 3 \\ \\ \\ 4 \\ \\ \end{array}$$

The global stiffness matrix:

$$
\mathbf{K}=\begin{array}{cc}
\begin{array}{cccccccccccc} & 1 & & & 2 & & & 3 & & & 4 & \end{array} & \\
\left[\begin{array}{ccc|ccc|ccc|ccc}
39787.2 & 39762.3 & -74.6 & -39787.2 & -39762.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
39762.3 & 39787.2 & 74.6 & -39762.3 & -39787.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-74.6 & 74.6 & 447.5 & 74.6 & -74.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-39787.2 & -39762.3 & 74.6 & 129787. & 39762.3 & 0 & -90000.0 & 0 & 0 & 0 & 0 & 0 \\
-39762.3 & -39787.2 & -74.6 & 39762.3 & 39931.2 & 360.0 & 0 & -144.0 & 360.0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 360.0 & 1200.0 & 0 & -360.0 & 600.0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -90000.0 & 0 & 0 & 90118.7 & 0 & 237.3 & -118.7 & 0 & 237.3 \\
0 & 0 & 0 & 0 & -144.0 & -360.0 & 0 & 84519.0 & -360.0 & 0 & -84375.0 & 0 \\
0 & 0 & 0 & 0 & 360.0 & 600.0 & 237.3 & -360.0 & 1832.8 & -237.3 & 0 & 316.4 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -118.7 & 0 & -237.3 & 118.7 & 0 & -237.3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -84375.0 & 0 & 0 & 84375.0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 237.3 & 0 & 316.4 & -237.3 & 0 & 632.8
\end{array}\right] &
\begin{array}{c} \\ 1 \\ \\ \\ 2 \\ \\ \\ 3 \\ \\ \\ 4 \\ \\ \end{array}
\end{array}
$$

The global stiffness matrix after taking into consideration boundary conditions:

$$\mathbf{K}^r = \begin{array}{c} \begin{array}{cccccccccccc} & 1 & & & 2 & & & 3 & & & 4 & \end{array} \\ \left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 129787. & 39762.3 & 0 & -90000.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 39762.3 & 39931.2 & 360.0 & 0 & -144.0 & 360.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 360.0 & 1200.0 & 0 & -360.0 & 600.0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -90000.0 & 0 & 0 & 90118.7 & 0 & 237.3 & 0 & 0 & 237.3 \\ 0 & 0 & 0 & 0 & -144.0 & -360.0 & 0 & 84519.0 & -360.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 360.0 & 600.0 & 237.3 & -360.0 & 1832.8 & 0 & 0 & 316.4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 237.3 & 0 & 316.4 & 0 & 0 & 632.8 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ \\ \\ 2 \\ \\ \\ 3 \\ \\ \\ 4 \\ \\ \end{array}$$

We calculate an unknown displacement (a rotation angle) of the cross section at node No 2 for element No 1 (φ_2^1) using the relation:

$$\mathbf{W}_{j\varphi} \cdot \mathbf{u}^1 + M_j^1 = 0$$

where

$$\mathbf{W}_{j\varphi} = \left[\begin{array}{ccc|ccc} -149.16 & 149.16 & 298.31 & 149.16 & -149.16 & 596.63 \end{array}\right]$$

is the sixth row of the stiffness matrix of the element No 1 $\mathbf{K}^1$ before condensation and

$$\mathbf{u}^1 = \left[\begin{array}{c} 0.0 \\ 0.0 \\ 0.0 \\ \hline 0.21315E-1 \\ -0.21679E-1 \\ \varphi_2^1 \end{array}\right]$$

is the displacement vector of element No 1; φ_2^1 is the unknown rotation of the cross section at the node and M_j^1 is the moment in this cross section.

In our example we have determined M_j^1 as the sixth component of the vector $\mathbf{f}^1$ before the condensation of $M_j^1 = -5.3032 \text{ kNm}$.

The solution of this equation gives

φ_2^1=-0.00186 rad.

Values of internal forces can be calculated from equation (4.10) by replacing nodal displacements contained in vector $\mathbf{u}$ by $\mathbf{u}^e$. Functions showing the variability of internal forces along the element axis can be determined from equations (4.18, 4.19, 4.20, 4.21, 4.22, 4.23). The results of calculations are presented in Tab.4.E1.3 and in the form of graphs (Fig.4.E1.2, Fig.4.E1.3, Fig.4.E1.4, Fig.4.E1.5).

Tab.4.E1.3

No e	*x/L* [/]	*Nu* [kN]	*Nz* [kN]	*N=Nu+Nz* [kN]	*Tu* [kN]	*Tz* [kN]	*T=Tu+Tz* [kN]	*Mu* [kNm]	*Mz* [kNm]	*M=Mu+Mz* [kNm]
1	0.00	-20.514	-7.500	-28.014	2.631	7.500	10.131	-5.858	-5.303	-11.162
	0.25	-20.514	-3.750	-24.264	2.631	3.750	6.381	-3.068	0.663	-2.405
	0.50	-20.514	0.000	-20.514	2.631	0.000	2.631	-0.277	2.652	2.374
	0.75	-20.514	3.750	-16.764	2.631	-3.750	-1.119	2.513	0.663	3.176
	1.00	-20.514	7.500	-13.014	2.631	-7.500	-4.869	5.303	-5.303	0.0
2	0.00	-12.646	0.0	-12.646	-2.990	8.750	5.760	6.250	-6.250	0.0
	0.25	-12.646	0.0	-12.646	-2.990	3.281	0.291	2.512	1.107	3.619
	0.50	-12.646	0.0	-12.646	-2.990	-0.625	-3.615	-1.226	2.604	1.378
	0.75	-12.646	0.0	-12.646	-2.990	-2.969	-5.959	-4.964	0.195	-4.769
	1.00	-12.646	0.0	-12.646	-2.990	-3.750	-6.740	-8.702	-4.167	-12.869
3	0.00	-9.039	0.0	-9.039	3.217	-7.5	-4.283	-7.500	7.5	0.0
	0.25	-9.039	0.0	-9.039	3.217	-7.5	-4.283	-4.283	0	-4.283
	0.50	-9.039	0.0	-9.039	3.217	-7.5	-4.283	-1.066	-7.5	-8.565
	0.50	-9.039	0.0	-9.039	3.217	7.5	10.717	-1.066	-7.5	-8.565
	0.75	-9.039	0.0	-9.039	3.217	7.5	10.717	2.152	0	2.152
	1.00	-9.039	0.0	-9.039	3.217	7.5	10.717	5.369	7.5	12.869

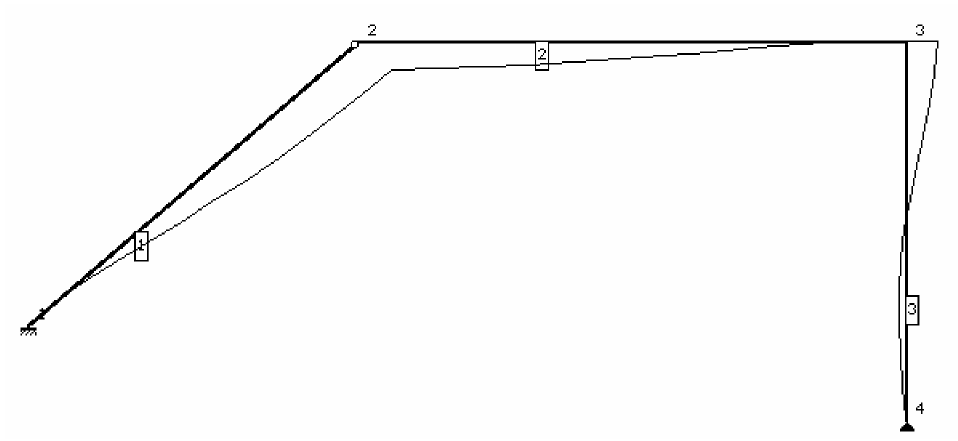

Fig.4.E1.2. The scheme of the deformed frame.

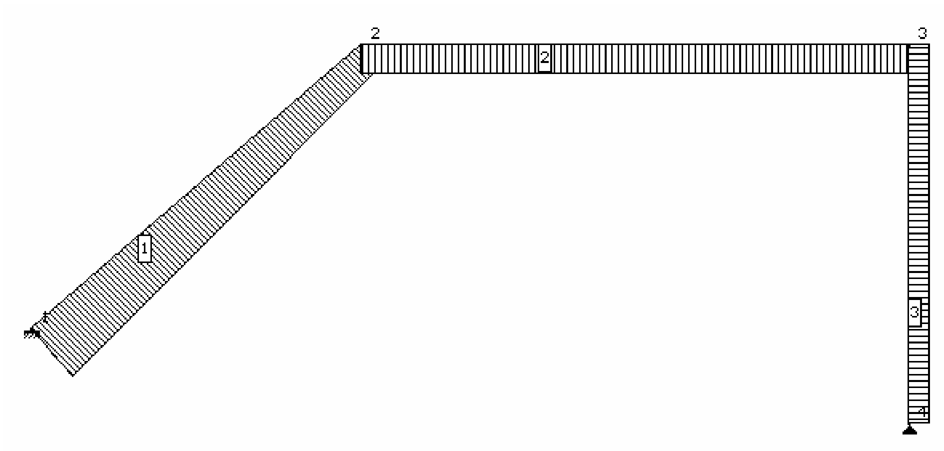

Fig.4.E1.3. The graph of normal forces.

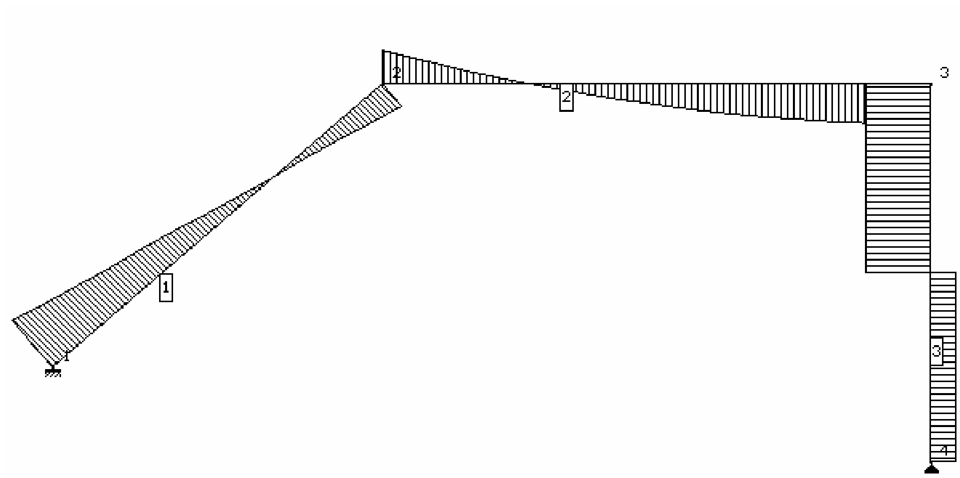

Fig.4.E1.4. The graph of shearing forces.

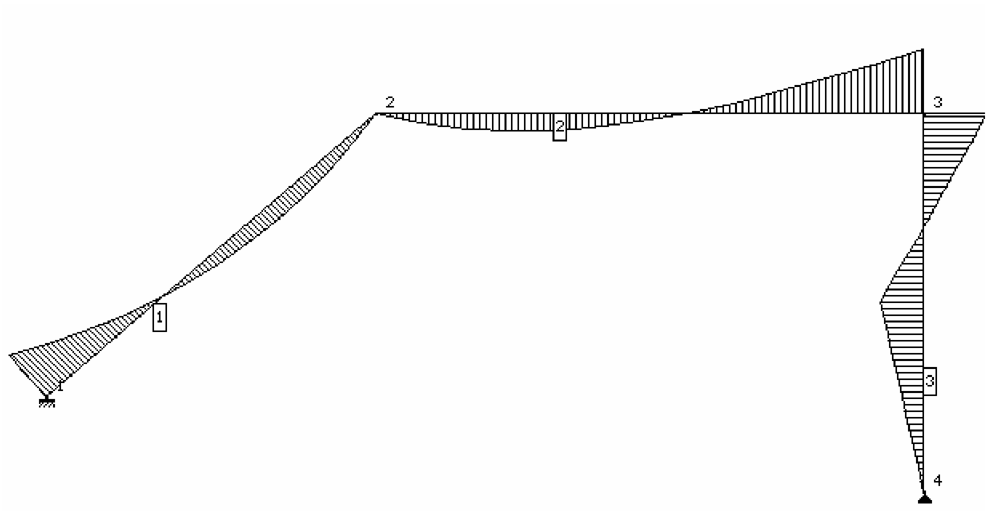

Fig.4.E1.5. The graph of bending moments.