CHAPTER I .
INTRODUCTION TO THE FINITE ELEMENT METHOD

In this chapter we will discuss basic conceptions and algorithms of the finite element method. We will also elaborate necessary information regarding the solid mechanics. As we have written in the Introduction, we assume that the reader knows basic issues of mechanics of materials and the theory of elasticity, therefore the discussed here information will be only a short survey and an introduction to the applied by us matrix notation. We recommend readers, who want to know more about it, the study of suitable references given at the end of this book, in particular books written by authors like S.Timoshenka [16],[17], D.N.Goodiera [17], Y.C.Funga [3] and P.Jastrzêbski et al [8].

1.1. THE ORIGIN AND BASIC CONCEPTION OF THE FINITE ELEMENT METHOD

We can find the beginnings of the finite element method in the '20s and '30s of the 20th century when authors like G.B.Maney and H.Cross in the USA and A.Ostenfeld in the Netherlands making use of findings presented in papers written by J. C. Maxwella, A. Castiliano and O. Mohra proposed a new method for solving structural mechanics problems which is now known as the displacement method.

In the middle of the 20th century J.Argyris, P.C.Pattan, S.Kelsey, M.Turner, R.Clough et al. accomplished the generalisation of this method. They did it on the basis of papers written by R.Couranta. In the '60s and '70s the finite element method was improved thanks to the publications written by O.C.Zienkiewicza, Y.K.Cheunga and R.L.Taylora. Thus it has become a contemporary tool used for solving issues of solid mechanics, temperature flows, fluid mechanics, electromagnetic fields and some other issues.

The basic idea of the finite element method (FEM) is searching for a solution of a complex problem (which is written in a form of a differential equation) by replacing it with a simpler and similar one. It most often leads to the discovery of an approximate solution, precision of which depends on assumed approximation methods. In mechanics problems a solution most often consists in determining displacements, strains and stresses of a continuum. These issues appear in statics and dynamics of frame constructions, plates, shells and solids. The equilibrium of a body is usually written in the form of a differential equation (or a set of differential equations) which has to be realised within the body and its boundary conditions

which should be realised on its surface. It is often very difficult or even impossible to find exact solutions. The finite element method proposes the following way of determination of an approximate solution [19]:

- The continuum is separated by imaginary lines or surfaces into a number of finite elements.
- The elements are assumed to be interconnected at a discrete number of nodal points situated on their boundaries. The displacements of these nodal points will be the basic unknown parameters of the problem, just as in the simple, discrete and structural analysis.
- A set of functions is chosen to define uniquely the state of displacement within each finite element in terms of its nodal displacements. The displacement functions now define uniquely the state of strain within an element in terms of the nodal displacements. These strains, together with any initial strains and the constitutive properties of the material, will define the state of stress throughout the element and, hence, also on its boundaries.
- Forces concentrated at the nodes (they are nodal forces) which are depended on nodal displacements is determined. The relationship between nodal forces and displacements is described the element stiffness matrix.
- A set of equilibrium equations is written for all nodes, hence the problem is done by solving a set of algebraical equations which are often linear ones. Solving such a set of equations with suitable boundary conditions allows to calculate strains and stresses within elements.

The approximation of the solution requires solving many problems of which the selection of shape functions and discrete systems seem to be the most important ones. The person choosing a construction model (elastic, plastic, frame, plate etc.) and a discrete method should have considerable experience. In the following chapters we will present necessary information to simplify the work of less experienced users of the finite element method systems.

1.2. BASIC ASSUMPTIONS AND THEOREMS OF SOLID MECHANICS

Here we will present a few basic assumptions and theorems of mechanics which will be used in the subsequent chapters of this book.

1.2.1. Assumptions regarding the linear model of a structure

In this chapter and some subsequent ones we will be dealing with linear problems of mechanics . It means that the process of structural deformation can be written by linear differential equations. It involves the following consequences:

- Displacements of structure points which appear during deformation are small. Linear displacements are considerably smaller than the characteristic dimension of a construction (for example, the deflection of a beam is a few hundreds times smaller than its length) and angles of rotation are considerably smaller than a unit (for example, a nodal angle of rotation is smaller than 0.01 rad).
- Strains are small, too. It allows to express the relationship between strains and displacements with the help of linear equations.
- The material is linear elastic which means that coefficients of constitutive equations describing stress-strain relationships are constant.

It may seem that such large limits which are put on both geometry of a construction and material characteristics strongly restrict the range of usage of the model. In effect these limits are realised for many structures (they can refer to most of them) created by people, so the range of usage of the model is very wide. The reader should know it when he proceeds with the description of any real problem of mechanics equations.

1.2.2. Stresses and strains

We will denote components of the stress tensor traditionally (as it occurs in most books on the finite element method), it means that components of direct stress will be denoted by letters σ_x, σ_y, σ_z and components of shear stress by τ_{xy}, τ_{xz}, τ_{yz}. Because of the symmetry of the stress tensor [17],[3] we will use only six components which composed in a column matrix form the stress vector:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}. \tag{1.1}$$

Denoting the components of the strain tensor traditionally we assume the following definitions:

$$\begin{aligned} &\varepsilon_x = \frac{\partial u_x}{\partial x}, \quad \varepsilon_y = \frac{\partial u_y}{\partial y}, \quad \varepsilon_z = \frac{\partial u_z}{\partial z}, \\ &\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \quad \gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}, \quad \gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \end{aligned} \tag{1.2}$$

where $\varepsilon_x, \varepsilon_y, \varepsilon_z$, are the components of direct strain (unit elongation) and $\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$ the components of shear strain(they are the angles of the non-dilatation strain), u_x, u_y, u_z are the components of the displacement vector in the cartesian coordinate system.

We write the components of strain in the form of a column matrix - the strain vector:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}. \tag{1.3}$$

We simplify the calculation of the internal work if we take the components of the strain vector γ_{ij} (the angles of the volumetric strain) instead of usual tensor definitions:

$$w = \int_V \sigma^T \varepsilon dV = \int_V \varepsilon^T \sigma dV \quad , \tag{1.4}$$

where V means the volume of a body.

1.2.3. Constitutive equations

As we have noted in our introductory assumptions, the relationship between the components of the stress tensor and the components of the strain tensor (that is, between $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ in our notation) is expressed by the linear equation:

$$\boldsymbol{\sigma} = \mathbf{D}\,\boldsymbol{\varepsilon}, \tag{1.5}$$

$$\boldsymbol{\varepsilon} = \mathbf{D}^{-1}\,\boldsymbol{\sigma}, \tag{1.6}$$

where $\mathbf{D}$ is the quadratic matrix with dimensions 6x6 containing the material constants:

$$\mathbf{D} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}, \tag{1.7}$$

where λ and μ are the Lamé constants.

Since some other material constants like Young's modulus - E and Poisson's ratio ν are more often used, in practice we present the relationships between them and the Lamé constants by the following formulae:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}. \tag{1.8}$$

The Lamé constant μ is noted by the letter G and is called Kirchhoff's modulus.

The inverse matrix of the matrix $\mathbf{D}^{-1}$ with the material constants has an unusually simple structure which is best shown by means of the constants E, ν:

$$\mathbf{D}^{-1} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}. \tag{1.9}$$

It should be noted that matrix **D** is symmetrical which means that the dependence $\mathbf{D}=\mathbf{D}^{T}$ occurs. This dependence will often be used in conversions.

1.2.4. Plane stress

In two-dimensional problems of thin plates the following simplification of the assumption is :

$$\sigma_z = 0, \tau_{zx} = 0, \tau_{zy} = 0, \tag{1.10}$$

which leads to the plane stress.

If we put equation (1.10) into equation (1.5) taking into consideration data from equation (1.7) we obtain:

$$\varepsilon_z = -\frac{\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right), \ \gamma_{zx} = 0, \ \gamma_{zy} = 0. \tag{1.11}$$

In plane stress the stress and strain vectors and the matrix of the material constants are reduced by half and thus:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}, \qquad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}, \tag{1.12}$$

$$\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}, \tag{1.13}$$

$$\mathbf{D}^{-1} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}. \tag{1.14}$$

1.2.5. Plane strain

In problems regarding deformations of massive buildings the plane strain is often found and it is expressed by the equations:

$$\varepsilon_z = 0, \gamma_{zx} = 0, \gamma_{zy} = 0, \tag{1.15}$$

When we insert the above equations into (1.6) taking also into consideration (1.9) we get the following relations:

$$\sigma_z = \nu\left(\sigma_x + \sigma_y\right), \quad \tau_{zx} = 0, \quad \tau_{zy} = 0 \tag{1.16}$$

After taking into consideration the above equations (1.15) and (1.16) we can notice that the relationship between the reduced stress and strain vectors (1.12) leads to the following matrix of elastic constants:

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}, \tag{1.17}$$

$$\mathbf{D}^{-1} = \frac{1-\nu^2}{E}\begin{bmatrix} 1 & \dfrac{-\nu}{1-\nu} & 0 \\ \dfrac{-\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \dfrac{2}{1-\nu} \end{bmatrix}. \tag{1.18}$$

1.2.6. Equilibrium equations

In order to keep a body fixed it is necessary that the external forces acting on this body should satisfy six equilibrium equations:

$$\sum_{i=1}^{n} \mathbf{P}_i = \mathbf{0}, \ \sum_{i=1}^{n} \mathbf{M}_i = \mathbf{0}, \tag{1.19}$$

which can be written as:

$$\sum_{i=1}^{n} P_{Xi} = \mathbf{0}; \qquad \sum_{i=1}^{n} P_{Yi} = \mathbf{0}; \qquad \sum_{i=1}^{n} P_{Zi} = \mathbf{0}; \tag{1.20}$$

$$\sum_{i=1}^{n} M_{Xi} = \mathbf{0}; \quad \sum_{i=1}^{n} M_{Yi} = \mathbf{0}; \quad \sum_{i=1}^{n} M_{Zi} = \mathbf{0},$$

where P_{Xi}, P_{Yi}, P_{Zi} are the components of the force $\mathbf{P}_i$ and M_{Xi}, M_{Yi}, M_{Zi} are the moments of this force in relation to the axes of a coordinate system and n is a number of forces.

When a set of forces is contained in, for example the plate XY, then equilibrium equations (1.20) are reduced to the following three equations:

$$\sum_{i=1}^{n} P_{Xi} = \mathbf{0}; \quad \sum_{i=1}^{n} P_{Yi} = \mathbf{0}; \quad \sum_{i=1}^{n} M_{Zi} = \mathbf{0}. \tag{1.21}$$

1.2.7. The principle of virtual work

Equilibrium equations (1.19) define conditions which a set of forces acting on a rigid body have to realise. In case of an elastic body which deforms by taking effect of acting forces we have to determine conditions for external forces, too. It can be done by using the principle of virtual work which says that the external work at virtual displacements is equivalent to the increase of the potential energy of internal forces:

$$\sum_{i=1}^{n} \mathbf{P}_i \cdot \overline{\mathbf{u}}_i = E_\sigma, \tag{1.22}$$

where $\overline{\mathbf{u}}_i$ is the vector of the virtual displacement at the point i, the dot means the scalar product of the vector of the force $\mathbf{P}_i$ and the vector of the virtual displacement $\overline{\mathbf{u}}_i$, E_σ - is the potential energy of internal forces:

$$E_\sigma = \int_V \boldsymbol{\sigma}^{\mathrm{T}} \overline{\boldsymbol{\varepsilon}} dV = \int_V \overline{\boldsymbol{\varepsilon}}^{\mathrm{T}} \boldsymbol{\sigma} dV. \tag{1.23}$$

In equation (1.23) $\overline{\boldsymbol{\varepsilon}}$ denotes the strain vector which appears during the virtual displacement $\overline{\mathbf{u}}$.

The virtual displacement ought to realise the following conditions [10]:

- it should be infinitely small,
- it should be independent of forces acting on a solid,
- it should be in accord with constraints taking a solid, so it is theoretically possible,
- it should be independent of time.

Equation (1.22) will be used many times in different forms in the subsequent chapters of this book.

1.2.8. Clapeyron's theorem

Changing virtual displacements into the real one in equations (1.22) and (1.23) we obtain:

$$\sum_{i=1}^{n} \mathbf{P}_i \cdot \mathbf{u}_i = \int_V \boldsymbol{\sigma}^{\mathrm{T}} \boldsymbol{\varepsilon} dV = \int_V \boldsymbol{\varepsilon}^{\mathrm{T}} \boldsymbol{\sigma} dV \,. \tag{1.24}$$

The above equation expresses the content of the Clapeyron's theorem which says that the work of external forces is equal to the potential energy of internal forces (elastic energy) for an elastic body being in equilibrium.

1.2.9. The Betti reciprocal theorem of work and the Maxwell reciprocal theorem of displacements

Let us insert the constitutive relation into equation (1.22) expressing the principle of virtual work (1.5). Thus we obtain:

$$\sum_{i=1}^{n} \mathbf{P}_i \cdot \overline{\mathbf{u}}_i = \int_V \boldsymbol{\sigma}^{\mathrm{T}} \overline{\boldsymbol{\varepsilon}} dV = \int_V (\mathbf{D}\boldsymbol{\varepsilon})^{\mathrm{T}} \overline{\boldsymbol{\varepsilon}} dV = \int_V \boldsymbol{\varepsilon}^{\mathrm{T}} \mathbf{D} \overline{\boldsymbol{\varepsilon}} dV \,. \tag{1.25}$$

In the above equations we have made use of the symmetry of the matrix of elastic constants $\mathbf{D}=\mathbf{D}^{\mathrm{T}}$.

Below we will apply the principle of virtual work in a different way, namely we attach virtual loads (a set of forces $\overline{\mathbf{P}}_j$) acting at the same nodes as the real loads do, but of a different value and direction. The work of these forces at the real displacement is equal to:

$$\sum_{i=1}^{n} \overline{\mathbf{P}}_j \cdot \mathbf{u}_j = \int_V \overline{\boldsymbol{\sigma}}^{\mathrm{T}} \boldsymbol{\varepsilon} dV = \int_V (\mathbf{D}\overline{\boldsymbol{\varepsilon}})^{\mathrm{T}} \boldsymbol{\varepsilon} dV = \int_V \overline{\boldsymbol{\varepsilon}}^{\mathrm{T}} \mathbf{D} \boldsymbol{\varepsilon} dV \,. \tag{1.26}$$

The right sides of equations (1.25) and (1.26) are identical which can be simply checked by direct calculations. Hence we obtain the equation:

$$\sum_{i=1}^{n} \mathbf{P}_i \cdot \overline{\mathbf{u}}_i = \sum_{i=1}^{n} \overline{\mathbf{P}}_i \cdot \mathbf{u}_i \,, \tag{1.27}$$

which expresses the reciprocal theorem of work formulated by E.Betti in 1972.

This theorem can be written as follows [10]:

The set of forces $\mathbf{P}_i$ makes the same work at the displacements, which are induced by the set of forces $\mathbf{P}_j$, as the set of forces $\mathbf{P}_j$ does at displacements which are induced by forces $\mathbf{P}_i$.

If we bring down both sets of forces to single unit forces acting at the point *a*, we obtain:

$$\mathbf{1}_a \cdot \overline{\mathbf{u}}_a = \overline{\mathbf{1}}_a \cdot \mathbf{u}_a \,. \tag{1.28}$$

This relationship is called the reciprocal theorem of displacements and was formulated by J.C.Maxwell in 1864.

1.3. ALGORITHM OF THE FINITE ELEMENT METHOD

The finite element method as a computer method is characterised by a strictly defined and simple algorithm. We will show the most important stages of this algorithm. Some of them will be discussed in detail in further parts of this book.

A. *Discretization*

At this stage the division of a structure into finite elements is done. In case of frame constructions it is often a natural division which means that every straight segment of a bar becomes an element. In case of surface structures we divide their area into triangular and quadrangular elements and in case of a solid construction we cut them into tetrahedral and hexahedronal elements.

At this stage we decide about places of contacts of elements, give coordinates of nodes and state the manner of connection between nodes and elements.

B. *Calculation of element stiffness matrices*

On the basis of material properties and topological data given in the first stage matrices expressing relationships between nodal forces and nodal displacements of an element are formed.

C. *Aggregation (construction) of a global stiffness matrix*

Now element stiffness matrices are divided into blocks which are put into a global stiffness matrix for which the information about construction topology is used. Modifications taking into consideration boundary conditions are often introduced into the global matrix at that stage.

D. *Construction of a global loads vector*

Here we calculate loads vectors of elements which after being divided into blocks are inserted into the global vector of nodal loads. When building the global vector is finished, then its components should be changed with regard to boundary conditions.

E. *Solution of a set of equations*

At this stage a set of linear equations will be solved. In effect we will obtain the nodal displacements of a structure.

F. *Calculation of internal forces and reactions*

If we calculate displacements, we can calculate strains, stresses and internal forces in a construction. After having calculated nodal forces of elements, reactions of constrains (supports) of the construction can be calculated, too.

The systems of FEM usually have a module structure. Individual stages of algorithm are realised by specialised modules of the system.

The first stage (A) complemented by given material properties and describing construction loads is called a preprocessor. In old systems this stage depended on manual creation of a data file (input file). At present such a situation occurs very rarely because introducing data for the typical problem of FEM (covering a few thousands of nodes) is very hard work. Contemporary preprocessors are usually graphic programmes equipped with tools simplifying generating of element meshes.

Stages (B), (C), (D), (E) are usually realised by the module called a processor. Apart from the mentioned above operations the processor often deals with a suitable arrangement of equations in order to reduce the amount of a necessary memory for storage of the stiffness matrix and to accelerate the process of solving necessary systems of equations.

The sixth stage (F) complemented by visual results is worked out by a postprocessor. A large amount of results that we get after solving any system of equations and calculating internal forces causes that analysing them without using visual techniques is very difficult. Contemporary systems of FEM are equipped with a graphic postprocessor drawing colour maps of stresses, displacements and other parameters which simplify analyses.

Although visual techniques are strongly joined with the finite element method, they are not a part of this method; hence they are not described in this book. We will concentrate only on these elements which are connected with calculations, that is, on the processor and parts of the postprocessor.

1.3.1. Creation of element stiffness matrices

As we have already noted in this chapter (point 1.1), we assume that after having divided the construction into finite elements, these elements are in contact with each other at nodes only. It will be convenient if we imagine a node as a material particle moving during the deformation process caused by external loads affecting the construction (forces, temperatures, etc.). We can describe the movement of a node by giving the components of displacement vectors. We will be interested in different types of motion according to the element type. In some cases they will be displacements (in truss elements, two-dimensional elements of a plate,

solids), in other cases they are rotations (in beams, frames, plates, shells), too. All necessary components of a nodal displacement create the system of parameters called degrees of freedom. We will mark the number of degrees of freedom as N_D.

In Tab.1.1 there is information on the number of degrees of freedom for nodes of typical engineering structures. Degrees of freedom are given as components of displacement vectors in the cartesian coordinate system.

Tab.1.1

Type of structure	Number of degrees of freedom N_D	Displacements			Rotations		
		u_x	u_y	u_z	φ_x	φ_y	φ_z
plane truss	2	•	•				
space truss	3	•	•	•			
plane frame	3	•	•				•
space frame	6	•	•	•	•	•	•
grillwork	3			•	•	•	
two-dimensional element	2	•	•				
plate	3			•	•	•	
shell	6	•	•	•	•	•	•
solid (brick)	3	•	•	•			

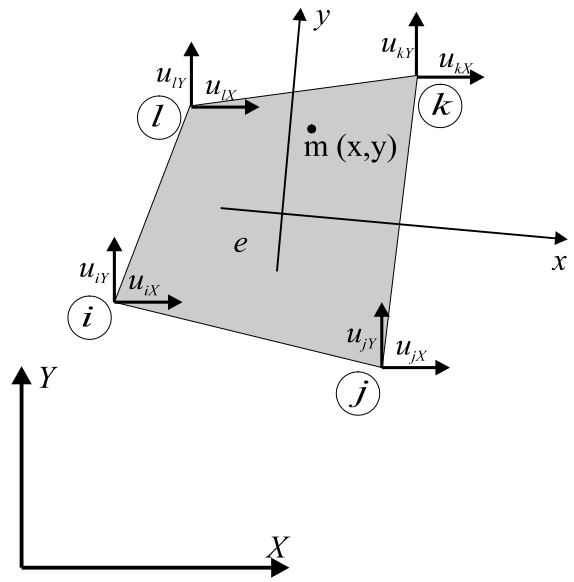

Fig.1.1

Let us imagine some quadrangular element (for convenience we will take a plane element which is easy to draw) having number e (Fig.1.1). The nodes of this element are locally numbered: i, j, k, l and they have their global numbers respectively: n_i, n_j, n_k, n_l. Nodal

coordinates are always given in the global coordinate system XY, but for convenience we use any local coordinate system while forming an element stiffness matrix. The local coordinate system is chosen at random.

We group nodal displacements in the displacement vector:

$$\mathbf{u}_i = \begin{bmatrix} u_{iX} \\ u_{iY} \end{bmatrix}, \ \mathbf{u}_j = \begin{bmatrix} u_{jX} \\ u_{jY} \end{bmatrix}, \ \mathbf{u}_k = \begin{bmatrix} u_{kX} \\ u_{kY} \end{bmatrix}, \ \mathbf{u}_l = \begin{bmatrix} u_{lX} \\ u_{lY} \end{bmatrix}. \tag{1.29}$$

The set of all nodal displacements of an element forms the vector of nodal displacements of an element:

$$\mathbf{u}^e = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \\ \mathbf{u}_k \\ \mathbf{u}_l \end{bmatrix} = \begin{bmatrix} u_{iX} \\ u_{iY} \\ u_{jX} \\ u_{jY} \\ u_{kX} \\ u_{kY} \\ u_{lX} \\ u_{lY} \end{bmatrix}. \tag{1.30}$$

The displacement of a certain point m being within the element is written in the form of the vector:

$$\mathbf{u}(X,Y) = \begin{bmatrix} u_X(X,Y) \\ u_Y(X,Y) \end{bmatrix}. \tag{1.31}$$

If the components of vectors are defined in a local coordinate system, then we will denote them as the sign ' (prim), for instance:

$$\mathbf{u}'(x,y) = \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix}. \tag{1.31a}$$

Similar notations can be used in equations (1.29) and (1.30) but for the time being we will use only global relationships for convenience.

Now we assume that the displacement of some point m can depend on nodal displacements of an element:

$$\mathbf{u}(x,y) = \mathbf{N}^e(x,y)\,\mathbf{u}^e, \tag{1.32}$$

where $\mathbf{N}(x,y)$ is the matrix component which depends on the coordinates of a point. The dimensions of the matrix $\mathbf{N}(x,y)$ depend on an element type. The number of rows of the matrix $\mathbf{N}(x,y)$ is equal to the number of degrees of freedom of the point m and the number of columns, that is, the number of degrees of freedom of elements. In our example where the point has two

degrees of freedom and the element has 4x2=8 degrees of freedom the matrix $\mathbf{N}(x,y)$ has to possess to two rows and eight columns.

Thus, it will be convenient to present equation (1.32) in a developed form:

$$\mathbf{u}(x,y) = \begin{bmatrix} \mathbf{N}_i(x,y) & \mathbf{N}_j(x,y) & \mathbf{N}_k(x,y) & \mathbf{N}_l(x,y) \end{bmatrix} \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \\ \mathbf{u}_k \\ \mathbf{u}_l \end{bmatrix}, \tag{1.32a}$$

where matrices $\mathbf{N}_i(x,y)$ ... $\mathbf{N}_l(x,y)$ are quadratic matrices containing functions which show the influence of the displacements of nodes *i* ... *l* on the displacement of the point *m*. In the finite element method these functions are known as *shape functions* or *displacement functions* and they are very important for the formulation of FEM equations. Matrices $\mathbf{N}_i(x,y)$ ... $\mathbf{N}_l(x,y)$ are called matrices of shape functions of nodes *i* ... *l* and the matrix $\mathbf{N}^e(x,y)$ is the matrix of shape functions of an element.

It is obvious that shape functions should fulfil some conditions which decide about their usefulness for the approximation of the field of an element displacement. If we imagine that the point *m* is at a node, then its displacements should be equal to the displacements of this node but the displacements of other nodes should not have any influence on them (Fig.1.2).

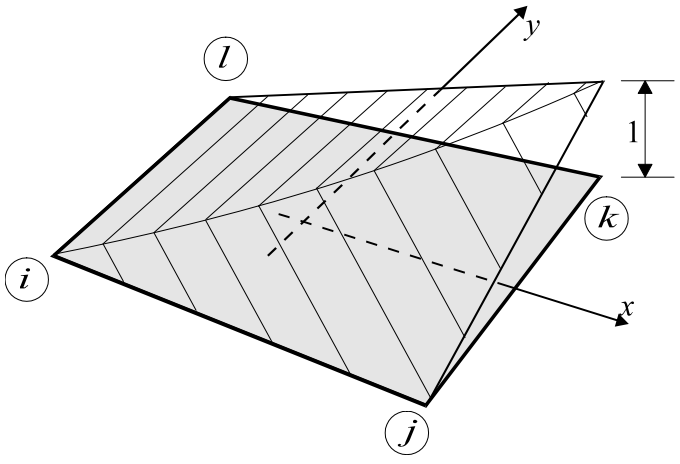

Fig.1.2

This condition can be expressed in the following way:

$$N_p(x_q, y_q) = \delta_{pq}, \tag{1.33}$$

where δ_{pq} is Kronnecker's delta:

$$\delta_{pq} = \begin{cases} 1 & \text{- when p=q,} \\ 0 & \text{- when p} \neq \text{q} \end{cases}$$

and p and q represent any local number of nodes i ... l.

Conditions of type (1.33) allow to determine the coefficients of shape functions. We will occupy with some other conditions which have to be fulfilled by functions $N_p(x,y)$ in further parts of this chapter.

Substituting equation (1.32) for (1.2) we calculate the components of the element strain vector:

$$\boldsymbol{\varepsilon} = \mathbf{D}\ \mathbf{N}^e(x,y)\mathbf{u}^e, \tag{1.34}$$

where $\mathbf{D}$ is the matrix with dimensions 3 x N_D for both plane stress and plane strain or 6 x N_D for three-dimensional problems (N_D is the number of degrees of freedom of a node) containing differential operators coming from the definition of strain (1.2).

For a two-dimensional problem N_D=2 the matrix of differential operators has the following form:

$$\mathbf{D} = \begin{bmatrix} \partial_x & 0 \\ 0 & \partial_y \\ \partial_y & \partial_x \end{bmatrix}, \tag{1.35}$$

where symbol ∂_x signifies differentiation in respect to x: $\partial_x = \dfrac{\partial}{\partial x}$ and ∂_y in respect to y.

We assume the notations:

$$\mathbf{D}\ \mathbf{N}^e(x,y) = \mathbf{B}^e(x,y) \tag{1.36}$$

and consistently

$$\begin{aligned} &\mathbf{D}\ \mathbf{N}_i(x,y) = \mathbf{B}_i(x,y), \\ &\vdots \\ &\mathbf{D}\ \mathbf{N}_l(x,y) = \mathbf{B}_l(x,y). \end{aligned} \tag{1.37}$$

They simplify further transformations.

After taking into consideration these notations, relation (1.34) can be presented as:

$$\boldsymbol{\varepsilon} = \mathbf{B}^e(x,y)\mathbf{u}^e, \tag{1.38}$$

The matrix $\mathbf{B}(x,y)$ has dimensions 3 (or 6 for three-dimensional problems of stress) x n_D^e.

For a quadrangular element of a two-dimensional problem matrix $\mathbf{B}(x,y)$ has dimensions 3x8. As we have done it with the matrix $\mathbf{N}(x,y)$, now we similarly divide the matrix $\mathbf{B}(x,y)$ into blocks:

$$\mathbf{B}^e(x,y) = \left[\mathbf{B}_i(x,y) \quad \mathbf{B}_j(x,y) \quad \mathbf{B}_k(x,y) \quad \mathbf{B}_l(x,y)\right]. \tag{1.39}$$

Matrices $\mathbf{B}_i$ ... $\mathbf{B}_l$ are matrices containing strain shape functions of nodes i ... l, and $\mathbf{B}^e(x,y)$ is the matrix containing strain shape functions of the element e.

Here we replace reactions between nodes and elements by concentrated forces. The scheme of these reactions is shown in Fig.1.3

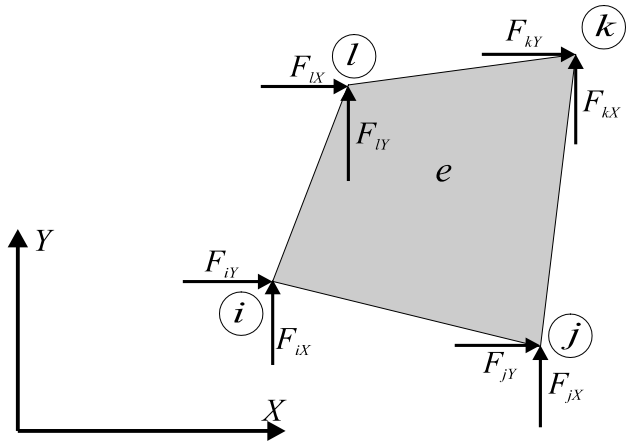

Fig.1.3

Now we collect the components of nodal forces into the nodal force vector:

$$\mathbf{f}_i = \begin{bmatrix} F_{iX} \\ F_{iY} \end{bmatrix}, \ \mathbf{f}_j = \begin{bmatrix} F_{jX} \\ F_{jY} \end{bmatrix}, \ \mathbf{f}_i = \begin{bmatrix} F_{kX} \\ F_{kY} \end{bmatrix}, \ \mathbf{f}_i = \begin{bmatrix} F_{lX} \\ F_{lY} \end{bmatrix}. \tag{1.40}$$

and the forces acting on an element into the nodal force vector of an element:

$$\mathbf{f}^e = \begin{bmatrix} \mathbf{f}_i \\ \mathbf{f}_j \\ \mathbf{f}_k \\ \mathbf{f}_l \end{bmatrix} = \begin{bmatrix} F_{iX} \\ F_{iY} \\ F_{jX} \\ F_{jY} \\ F_{kX} \\ F_{kY} \\ F_{lX} \\ F_{lY} \end{bmatrix}. \tag{1.41}$$

Let us look for the relation between nodal forces $\mathbf{f}^e$ and nodal displacements $\mathbf{u}^e$.

We apply the principle of virtual work (1.22) treating the nodal forces as the external loads of an element. The element is loaded both on its inside and boundary and we denote the load which depends on the coordinates of a point as follows:

$$\mathbf{q}(x,y)=\begin{bmatrix} q_x(x,y) \\ q_y(x,y) \end{bmatrix}. \tag{1.42}$$

We supplement constitutive equation (1.5) (or for instance (1.12) and (1.13) for plane stress) with parts allowing to consider initial strains and stresses:

$$\boldsymbol{\sigma}=\mathbf{D}\left(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_o\right)+\boldsymbol{\sigma}_o, \tag{1.43}$$

where $\boldsymbol{\varepsilon}_o$ is the initial strain vector (for example, caused by temperature loads) and $\boldsymbol{\sigma}_o$ is the initial stress vector.

Now we are writing equation (1.22) expressing the equality of external and internal work for the element in equilibrium:

$$\left(\mathbf{u}^e\right)^{\mathrm{T}}\mathbf{f}^e+\int_{\mathrm{A}}\mathbf{u}(x,y)^{\mathrm{T}}\mathbf{q}(x,y)d\mathrm{A}=\int_{\mathrm{V}}\varepsilon^{\mathrm{T}}\sigma d\mathrm{V}\ . \tag{1.44}$$

The left side of this equation presents external work while the right side denotes internal work for this element, **A** represents the surface of an element and **V** is its volume. We use (1.32), (1.38) and (1.43) in the above equation:

$$\left(\mathbf{u}^{\mathrm{e}}\right)^{\mathrm{T}}\mathbf{f}^{\mathrm{e}}+\int_{\mathrm{A}}\left(\mathbf{N}^{\mathrm{e}}\mathbf{u}^{\mathrm{e}}\right)^{\mathrm{T}}\mathbf{q}\mathrm{dA}=\int_{\mathrm{V}}\left(\mathbf{B}^{\mathrm{e}}\mathbf{u}^{\mathrm{e}}\right)^{\mathrm{T}}\left[\mathbf{D}\left(\mathbf{B}^{\mathrm{e}}\mathbf{u}^{\mathrm{e}}-\varepsilon_{\mathrm{o}}\right)+\sigma_{\mathrm{o}}\right]\mathrm{dV}\ . \tag{1.45}$$

After the transformation we obtain its final form as follows:

$$\mathbf{f}^e=\mathbf{K}^e\mathbf{u}^e-\mathbf{f}_q^e-\mathbf{f}_{\varepsilon_o}^e+\mathbf{f}_{\sigma_o}^e. \tag{1.46}$$

where the following values have been noted:

- nodal forces vector due to external loads:

$$\mathbf{f}_q^e=\int_{\mathrm{A}}\left(\mathbf{N}^e\right)^{\mathrm{T}}\mathbf{q}\,d\mathrm{A}\ , \tag{1.47}$$

- nodal forces vector due to initial strain:

$$\mathbf{f}_{\varepsilon_o}^e=\int_{\mathrm{V}}\left(\mathbf{B}^e\right)^{\mathrm{T}}\mathbf{D}\varepsilon_o d\mathrm{V}\ , \tag{1.48}$$

- nodal forces vector due to initial stress:

$$\mathbf{f}_{\sigma_o}^e=\int_{\mathrm{V}}\left(\mathbf{B}^e\right)^{\mathrm{T}}\sigma_o d\mathrm{V}\ , \tag{1.49}$$

- element stiffness matrix:

$$\mathbf{K}^e = \int_V \left(\mathbf{B}^e\right)^{\mathrm{T}} \mathbf{D}\,\mathbf{B}^e dV \quad . \tag{1.50}$$

Thus calculated nodal forces vectors contain forces acting on the element. They ought to be marked with the negative sign while performing equilibrium equations.

The matrix $\mathbf{K}^e$ can be divided into a block of quadratic matrices $\mathbf{K}^e_{pq}$ describing the influence of the displacement of the node *q* on the forces at the node *p*:

$$\mathbf{K}^e_{pq} = \int_V \left(\mathbf{B}^e_p\right)^{\mathrm{T}} \mathbf{D}\,\mathbf{B}^e_q dV \quad . \tag{1.51}$$

There are 4x4=16 blocks in the stiffness matrix of the element with four nodes (Fig.1.3). Since the stiffness matrix is symmetrical, it means that $\mathbf{K}^e = \left(\mathbf{K}^e\right)^{\mathrm{T}}$ which comes from equation (1.50) and it is a simple consequence of the Betti reciprocal theorem of work; then blocks $\mathbf{K}^e_{pq}$ have to realise the conditions:

$$\mathbf{K}^e_{qp} = \left(\mathbf{K}^e_{pq}\right)^{\mathrm{T}} . \tag{1.52}$$

Equation (1.50) or (1.51) constitutes a key step in formulating equilibrium equations of the structure but the stiffness matrix has not always been determined in this way. For simple elements such as a truss element or a frame element some other ways (sometimes simpler) of obtaining relation (1.46) exist. We will show the use of these ways in next chapters.

If all transformations leading to equation (1.50) have been done in the local coordinate system (*xyz*), then the obtained stiffness matrix should be transformed to the global coordinate system (*XYZ*). This transformation depends on multiplication of the matrix $\mathbf{K}^{e\prime}$ (sign prim denotes a matrix in the local coordinate system) by the transformation matrix of the element. The detailed structure of these matrices is elaborated in chapters II, III i IV, here we show the idea of transformation only:

$$\mathbf{K}^e = \mathbf{R}^e \mathbf{K}'^e \left(\mathbf{R}^e\right)^{\mathrm{T}}, \tag{1.53}$$

$$\text{where } \mathbf{R}^e = \begin{bmatrix} \mathbf{R}_i & & & \\ & \mathbf{R}_j & & \\ & & \mathbf{R}_k & \\ & & & \ddots \end{bmatrix}, \tag{1.54}$$

$\mathbf{R}_i$ ... $\mathbf{R}_k$ - transformation matrices of nodes *i* ... *k*. The transformation matrices of the nodes contain cosines of angles between axes of the global and local coordinate systems:

$$\mathbf{R}_i = \begin{bmatrix} C_{xX} & C_{xY} & C_{xZ} \\ C_{yX} & C_{yY} & C_{yZ} \\ C_{zX} & C_{zY} & C_{zZ} \end{bmatrix}, \tag{1.55}$$

where for instance $C_{xY} = \cos(\alpha_{xY})$, etc., α_{xY} is the angle between the x axis of the local coordinate system and the Y axis of the global system.

1.3.2. Aggregation (construction) of a global stiffness matrix

Relation (1.46) allows to write equilibrium equations of a node in the form containing nodal displacements as unknown.

Let us imagine a node as an independent part of a construction and disconnect elements from nodes in order to show nodal forces (Fig.1.4).

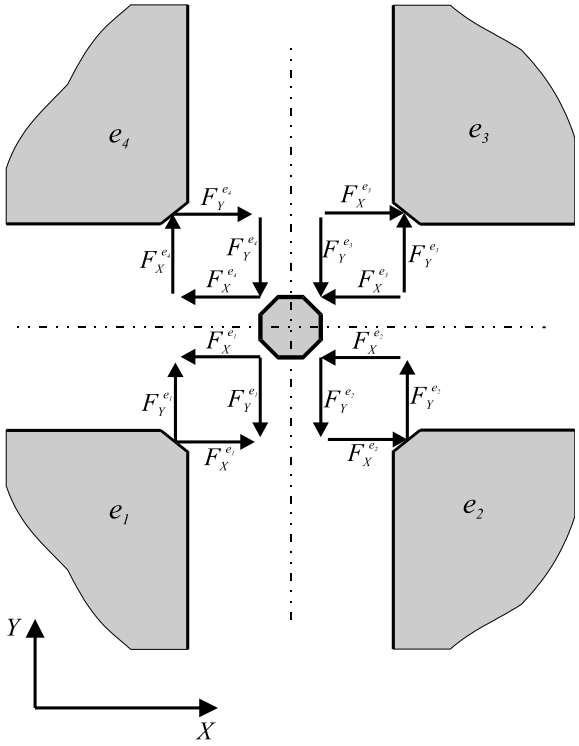

Fig.1.4

We write a set of equilibrium equations of the node in the scalar form:

$$\sum_{k=1}^{E_n} F_X^{e_k} = 0 \ , \ \sum_{k=1}^{E_n} F_Y^{e_k} = 0 \ , \ \sum_{k=1}^{E_n} F_Z^{e_k} = 0 \ , \tag{1.56a}$$

For the nodes with the rotation degrees of freedom the equilibrium equations of moments will be necessary:

$$\sum_{k=1}^{E_n} M_X^{e_k} = 0 \ , \ \sum_{k=1}^{E_n} M_Y^{e_k} = 0 \ , \ \sum_{k=1}^{E_n} M_Z^{e_k} = 0 \ . \tag{1.56b}$$

In equations (1.56) summation is done for all elements connected to the node, hence indices e_1, e_2 ... e_{E_n} are numbers of elements connected to the node, E_n is the number of elements connected to the node n. We insert relationship (1.46) into equations (1.56) remembering about changing a sign of the nodal forces coming from the change of sense of the forces acting on the element and node (Fig.1.4):

$$-\sum_{k=1}^{E_n} \mathbf{f}_n^{e_k} = 0 . \tag{1.57}$$

In this equation symbol $\mathbf{f}_n^{e_k}$ defines only these components of vector $\mathbf{f}^{e_k}$ which act on the node n. We convert this equation into a more convenient form:

$$\sum_{k=1}^{E_n} \mathbf{K}_n^{e_k} \mathbf{u}_n^{e_k} = \mathbf{p}_n^{e_k} , \tag{1.58}$$

where $\mathbf{p}^e = \mathbf{f}_q^e + \mathbf{f}_{\varepsilon_o}^e - \mathbf{f}_{\sigma_o}^e$ is the vector of the nodal forces due to external loads, initial strains and stresses.

Arranging for every node of the construction equations similar to equation (1.56), we obtain a set of equations which allow to calculate nodal displacements of this construction. Since summation is done for the elements in equation (1.56) (the force vectors which belong to this node), formation of a set of equations based on balancing of successive nodes is not effective.

Ordering nodes and degrees of freedom is necessary for this operation. So far we have used local numbers of nodes of elements i, j, k, l .., but introducing global numeration of nodes is necessary while building the global set of equations. Let n_i stand for a global number of the node represented by the local number i and let s_p be a global number of degrees of freedom represented by the local number p. Now we form a rectangular matrix of connections of the element e - $\mathbf{A}^e$. The number of rows of the matrix $\mathbf{A}^e$ is equal to the global number of degrees of freedom of the construction N_k, the number of columns is equal to the number of degrees of freedom of the element e - N_D^e. Most components of the matrix $\mathbf{A}^e$ is equal to zero apart from the components having the value of 1 which are situated in rows s_p and columns p. Hence the structure of the matrix $\mathbf{A}^e$ contains information about connections between the element and nodes or being more exact about the correspondence between the

degree of freedom of the element and the global degree of freedom of the construction. The formation of the connection matrix can be most easily studied on the following example.

Fig.1.5 presents a plate divided into five triangular elements. The plate has six nodes numbered from 1 to 6, every element has a local notation of nodes *i*, *j*, *k*. Tab.1.2 shows global numeration of degrees of freedom of a two-dimensional element of the plate.

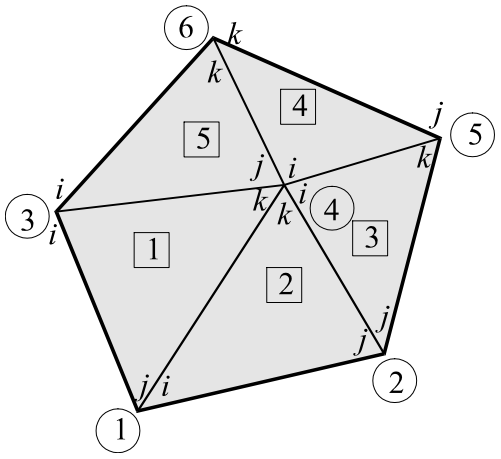

Fig.1.5

Tab.1.2

Node number n	Global numbers of degrees of freedom of nodes	
	u_{nX}	u_{nY}
1	1	2
2	3	4
3	5	6
4	7	8
5	9	10
6	11	12

Tab.1.3

Element number e	Global numbers of degrees of freedom of element s_p - allocation vector					
	u_{iX}	u_{iY}	u_{jX}	u_{jY}	u_{kX}	u_{kY}
	1	2	3	4	5	6
1	5	6	1	2	7	8
2	1	2	3	4	7	8
3	7	8	3	4	9	10
4	7	8	9	10	11	12
5	5	6	7	8	11	12

Tab.1.3 shows the dependence between local and global degrees of freedom. Hence the connection matrix created for element No 3 will have the following form:

$$\mathbf{A}^3=\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} & & & & & \\ & & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & & \\ & & & & & \\ 1 & & & & & \\ & 1 & & & & \\ & & & & 1 & \\ & & & & & 1 \\ & & & & & \\ & & & & & \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{matrix}$$

where all zero elements are neglected for clarity.

Multiplying the nodal force vector of an element by the connection matrix causes the transfer of suitable blocks of the local vector to the global vector. Now a simple addition of these vectors is possible:

$$\sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{f}^e = \sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{K}^e \mathbf{u}^e = \sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{p}^e \,. \tag{1.59}$$

Here it is necessary to express the nodal displacement vector of elements by means of the global vector:

$$\mathbf{u}^e = \left(\mathbf{A}^e\right)^{\mathrm{T}} \mathbf{u}\,,$$

which should be put into equation (1.59). Finally, we obtain the system of equations in the form like that:

$$\sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{K}^e \left(\mathbf{A}^e\right)^{\mathrm{T}} \mathbf{u} = \sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{p}^e \,, \tag{1.60}$$

or in a shorter form

$$\mathbf{K}\,\mathbf{u} = \mathbf{p}\,. \tag{1.61}$$

Matrix $\mathbf{K} = \sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{K}^e \left(\mathbf{A}^e\right)^{\mathrm{T}}$ is called the global stiffness matrix of a structure, vector $\mathbf{p} = \sum_{e=1}^{N_E} \mathbf{A}^e \mathbf{p}^e$ is the global vector of nodal forces of the structure, the vector **u** containing the displacement of all nodes is the global displacement vector.

A similar method of aggregation is described in the book written by G. Rakowski [12] where matrix $\mathbf{A}^{\mathrm{T}}$ is called the adhesion matrix.

The method of aggregation using the adhesion matrix is not suitable for computer implementation because it uses the big matrix $\mathbf{A}^e$. It is more effective to exploit information which is contained in allocation vectors. Vectors for the discussed example are included in Tab.1.3. The aggregation method using allocation vectors will be presented in the second chapter in parts devoted to building the stiffness matrix of a truss.

1.3.3. Remarks regarding the shape functions of an element

Functions approximating the displacement field within elements which are in fact shape functions described in Sec.1.3.1 cannot be chosen in a free way. They should be fulfilled some conditions which decide about the quality of these functions or their usefulness for

approximation of displacements, strains and stresses. We quote these criteria after O.C.Zienkiewicz [19].

A. *Criteria of rigid body movements*

The displacement function chosen should be in such a way that it should not permit straining of an element to occur when the nodal displacements are caused by a rigid body displacement.

B. *Criterion of strain stability*

The displacement function has to be of such a form that if nodal displacements are compatible with a constant strain condition such constant strain will in fact be obtained.

C. *Criterion of strain agreement*

The displacement functions should be so chosen that the strains at the interface between elements are finite.

Criteria (A) and (B) seem to be obvious. Since strain and stress causing some components of strain (or stress) of the zero value are possible, then approximation functions should be utterly able to reproduct these problems. Constant and linear parts of polynomials which we most often use to build a shape structure assure realisation of conditions (A) and (B). Criterion (B) is the generalisation of criterion (A) and it was formulated by Bazeley, Cheung, Irons and Zienkiewicz [19,20] in 1965.

Criterion (C) requires that shape functions should assure continuity of derivatives to the degree which is lower by one than differential operators being in the matrix **D** (comp. equation (1.34)). We explain it on the following example. In the two-dimensional problem of a plate the strains are defined by the first derivation of the displacement function (comp. (1.34) and (1.35)) because the displacement field has to be continuous on the boundary between elements and displacements functions have to be of class C^0. For plate elements the curvatures given by the second order derivatives take the role of displacements (comp. chapter VII). Hence the displacement function of a plate should assure continuity both of the surface of a plate deflection and its first derivations inside and on the boundaries between elements. Then the displacement field should be continuous and smooth within the plate. These functions are said to be of class C^1.

Criteria (A) and (B) have to be necessarily realised, criterion (C) does not, for instance, the shape function of plate elements does not often realise the condition of continuity (continuity of the first derivations on boundaries of elements). If all criteria are realised, then

we say that described elements are adjust ones. If only criteria (A) and (B) are realised, then elements are called not adjust ones.

The result of applying adjust and not adjust elements to discretization of a structure is presented in Fig.1.6. The convergence of results obtained with the help of the different types of elements which are used for discretization of a quadratic plate is shown in the same figure.

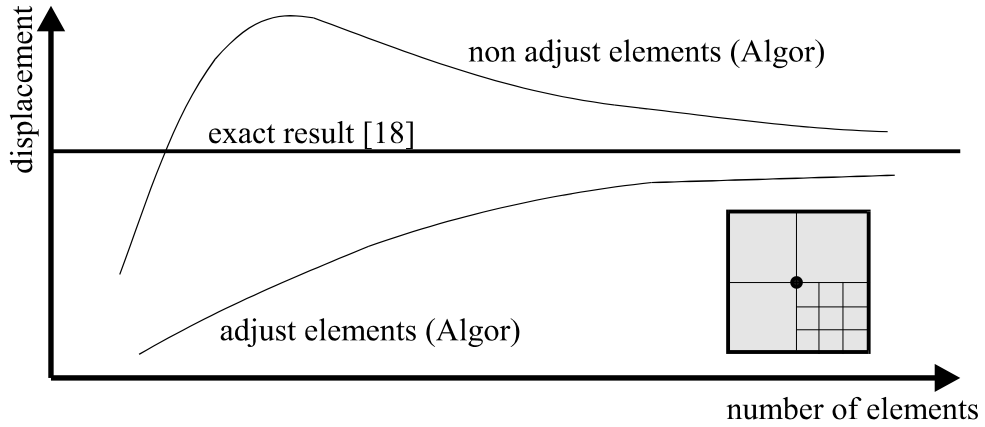

Fig.1.6

Apart from the three listed criteria we can also add some other ones which determine the choice of approximation polynomials. This choice should assure isotropy in respect to axes of a coordinate system. We will show it on the example of building shape functions of plate elements (two- and three-dimensional problems). If we present approximation polynomials in the form of Pascal triangle, then the choice of part of this triangle should be symmetrical in respect to its axes. It is shown in Fig.1.7.

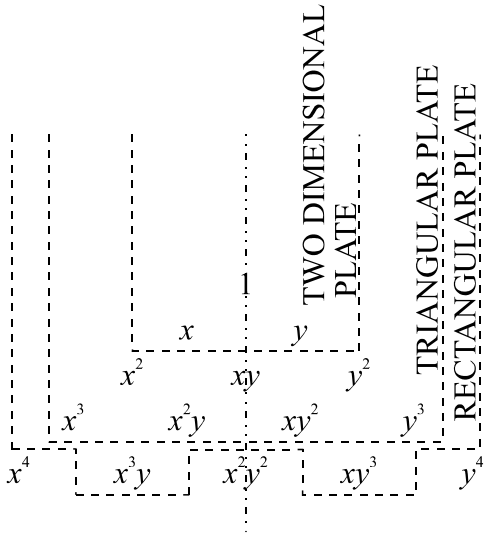

Fig.1.7

We can use Hermitte (described in chapter IV of this book) and Lagrange polynomials [19], too, but we always have to keep the condition of isotropy.

There is a long list of references as far as shape functions are concerned but we recommend the following books [1, 12, 13, 20].