Example 2.E1.

The content of the example

Fig.2.E1.1 shows a 2D truss made from wood (Young's modulus $E=1.2\cdot10^7$ kPa), loaded with concentrated forces acting on two nodes. The truss is composed of elements (bars) with three different cross sections. The bars of a bottom flange (elements No 1, 2, 3) have the cross section A-A, the bars of a top flange (elements No 4, 8, 9) have the cross section B-B, the cross-braces and the posts (elements No 5, 6, 7) have the cross section C-C.

Determine the global stiffness matrix of this truss, the vector of global nodal forces, internal forces and stresses in the elements.

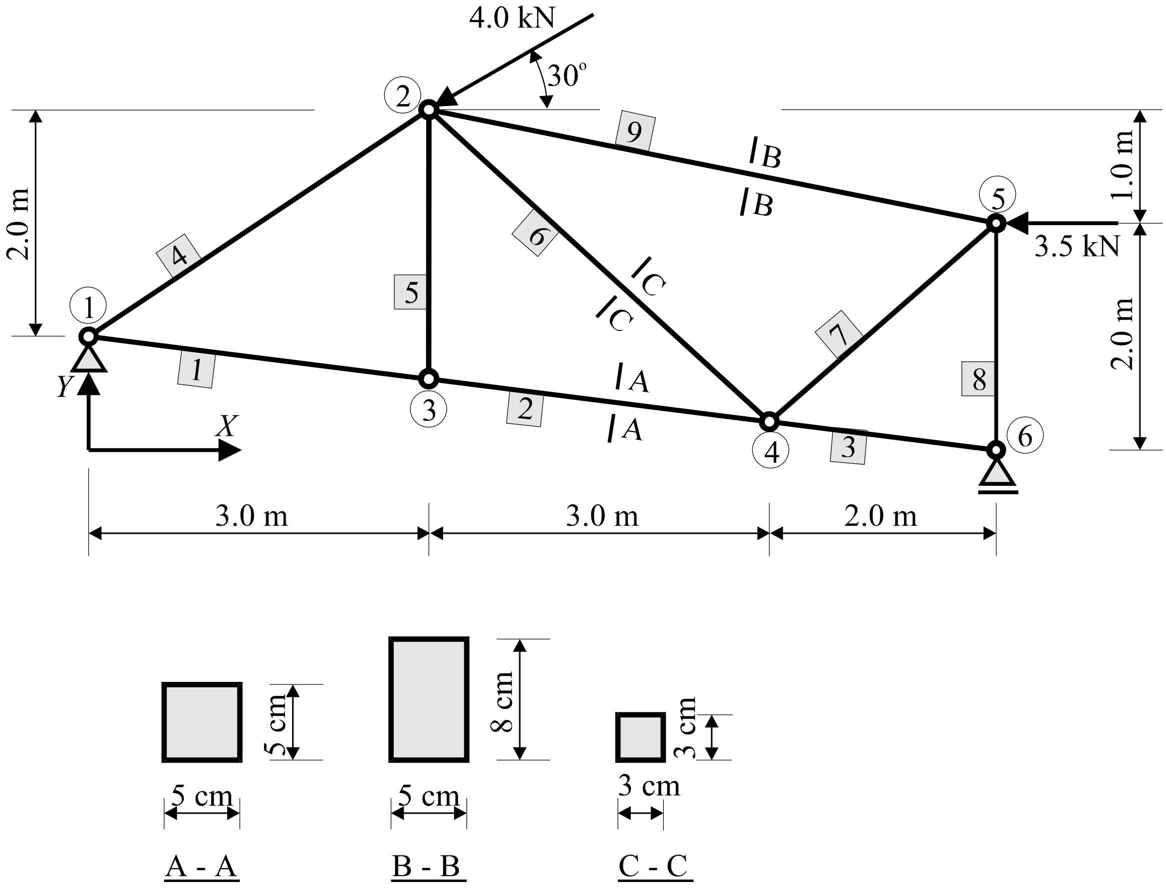

Fig.2.E1.1

Data concerning nodal coordinates and the load of the truss are collected in Tab.2.E1.1. Data concerning elements like node numbers (n_i, n_j), the projections of the elements on the axes of the global coordinate system (L_x, L_y), the angles of inclination with regard to the axis (α) and the surface of the cross section (A) are given in Tab.2.E1.2.

Tab.2.E1.1

Node No n	X_n [m]	Y_n [m]	P_{Xn} [kN]	P_{Yn} [kN]
1	0.0	1.0	------	------
2	3.0	3.0	-3.464	-2.0
3	3.0	0.625	------	------
4	6.0	0.25	------	------
5	8.0	2.0	-3.5	------
6	8.0	0.0	------	------

Tab.2.E1.2

Elem. No n	Node No n_i	n_j	L_{nX} [m]	L_{nY} [m]	L_n [m]	α_n [deg]	Section No	A [m^2]
1	1	3	3.0	-0.375	3.02335	-7.125	A-A	$2.5 \cdot 10^{-3}$
2	3	4	3.0	-0.375	3.02335	-7.125	A-A	$2.5 \cdot 10^{-3}$
3	4	6	2.0	-0.25	2.01556	-7.125	A-A	$2.5 \cdot 10^{-3}$
4	1	2	3.0	2.0	3.60555	33.690	B-B	$4.0 \cdot 10^{-3}$
5	3	2	0.0	2.375	2.375	90.000	C-C	$9.0 \cdot 10^{-4}$
6	2	4	3.0	-2.75	4.06971	-42.510	C-C	$9.0 \cdot 10^{-4}$
7	4	5	2.0	1.75	2.65754	41.186	C-C	$9.0 \cdot 10^{-4}$
8	6	5	0.0	2.0	2.0	90.000	B-B	$4.\ 0 \cdot 10^{-3}$
9	2	5	5.0	-1.0	5.09902	-11.310	B-B	$4.0 \cdot 10^{-3}$

The solution of the example

We start solving the problem from building the vector of global nodal forces, which is very simple when the truss is loaded with concentrated forces only. The data juxtaposed in Tab.2.E1.2 are applied to building the vector **p**:

$$
\mathbf{p}=\begin{array}{c|c|c}
 & \text{Global node number} & \text{Global number of degree of freedom} \\
\left[\begin{array}{c} 0.0 \\ 0.0 \\ \hline -3.4641 \\ -2.0 \\ \hline 0.0 \\ 0.0 \\ \hline 0.0 \\ 0.0 \\ \hline -3.5 \\ 0.0 \\ \hline 0.0 \\ 0.0 \end{array}\right] &
\begin{array}{c} 1 \\ \\ 2 \\ \\ 3 \\ \\ 4 \\ \\ 5 \\ \\ 6 \\ \\ \end{array} &
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array}
\end{array}
$$

Since no concentrated forces load a support node, the vector of global nodal forces, after taking into consideration boundary conditions, is identical

$\mathbf{p}^r = \mathbf{p}$.

Now we form element stiffness matrices of the truss. As we showed in Chapter II, element stiffness matrices are build of the matrices $\mathbf{J}^e$ with dimensions 2x2 (comp. (2.37)). On the basis of equation (2.37) we determine

$$
\mathbf{J}^1=\begin{bmatrix} 9770.12 & -1221.27 \\ -1221.27 & 152.66 \end{bmatrix} \qquad \mathbf{J}^2=\begin{bmatrix} 9770.12 & -1221.27 \\ -1221.27 & 152.66 \end{bmatrix}
$$

$$\mathbf{J}^3=\begin{bmatrix} 14655.18 & -1831.90 \\ -1831.90 & 228.99 \end{bmatrix} \qquad \mathbf{J}^4=\begin{bmatrix} 9216.56 & 6144.37 \\ 6144.37 & 4096.25 \end{bmatrix}$$

$$\mathbf{J}^5=\begin{bmatrix} 0.00 & 0.00 \\ 0.00 & 4547.37 \end{bmatrix} \qquad \mathbf{J}^6=\begin{bmatrix} 1442.04 & -1321.87 \\ -1321.87 & 1211.71 \end{bmatrix}$$

$$\mathbf{J}^7=\begin{bmatrix} 2301.69 & 2013.98 \\ 2013.98 & 1762.23 \end{bmatrix} \qquad \mathbf{J}^8=\begin{bmatrix} 0.00 & 0.00 \\ 0.00 & 24000.00 \end{bmatrix}$$

$$\mathbf{J}^9=\begin{bmatrix} 9051.51 & -1810.30 \\ -1810.30 & -1810.30 \end{bmatrix}$$

We form the stiffness matrix from the following blocks:

$$\mathbf{K}=\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{c|c|c|c|c|c} \mathbf{J}^1+\mathbf{J}^4 & -\mathbf{J}^4 & -\mathbf{J}^1 & & & \\ \hline -\mathbf{J}^4 & \mathbf{J}^4+\mathbf{J}^5+\mathbf{J}^6+\mathbf{J}^9 & -\mathbf{J}^5 & -\mathbf{J}^6 & -\mathbf{J}^9 & \\ \hline -\mathbf{J}^1 & -\mathbf{J}^5 & \mathbf{J}^1+\mathbf{J}^2+\mathbf{J}^5 & -\mathbf{J}^2 & & \\ \hline & -\mathbf{J}^6 & -\mathbf{J}^2 & \mathbf{J}^2+\mathbf{J}^3+\mathbf{J}^6+\mathbf{J}^7 & -\mathbf{J}^7 & -\mathbf{J}^3 \\ \hline & -\mathbf{J}^9 & & -\mathbf{J}^7 & \mathbf{J}^7+\mathbf{J}^8+\mathbf{J}^9 & -\mathbf{J}^8 \\ \hline & & & -\mathbf{J}^3 & -\mathbf{J}^8 & \mathbf{J}^3+\mathbf{J}^8 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

and after substituting previously obtained matrices $\mathbf{J}^e$, we obtain all components of the matrix **K** which are tabulated on the next page.

The global stiffness matrix:

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \quad\quad 1 \quad\quad & \quad\quad 2 \quad\quad & \quad\quad 3 \quad\quad & \quad\quad 4 \quad\quad & \quad\quad 5 \quad\quad & \quad\quad 6 \quad\quad \end{array} \\
\mathbf{K}= & \left[\begin{array}{cc|cc|cc|cc|cc|cc}
18986.7 & 4923.1 & -9216.6 & -6144.4 & -9770.1 & 1221.3 & 0 & 0 & 0 & 0 & 0 & 0 \\
4923.1 & 4248.9 & -6144.4 & -4096.2 & 1221.3 & -152.7 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-9216.6 & -6144.4 & 19710.1 & 3012.2 & 0 & 0 & -1442.0 & 1321.9 & -9051.5 & 1810.3 & 0 & 0 \\
-6144.4 & -4096.2 & 3012.2 & 10217.4 & 0 & -4547.4 & 1321.9 & -1211.7 & 1810.3 & -362.1 & 0 & 0 \\
\hline
-9770.1 & 1221.3 & 0 & 0 & 19540.2 & -2442.5 & -9770.1 & 1221.3 & 0 & 0 & 0 & 0 \\
1221.3 & -152.7 & 0 & -4547.4 & -2442.5 & 4852.7 & 1221.3 & -152.7 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -1442.0 & 1321.9 & -9770.1 & 1221.3 & 28169.0 & -2361.1 & -2301.7 & -2014.0 & -14655.2 & 1831.9 \\
0 & 0 & 1321.9 & -1211.7 & 1221.3 & -152.7 & -2361.1 & 3355.6 & -2014.0 & -1762.2 & 1831.9 & -229.0 \\
\hline
0 & 0 & -9051.5 & 1810.3 & 0 & 0 & -2301.7 & -2014.0 & 11353.2 & 203.7 & 0 & 0 \\
0 & 0 & 1810.3 & -362.1 & 0 & 0 & -2014.0 & -1762.2 & 203.7 & 26124.3 & 0 & -24000.0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -14655.2 & 1831.9 & 0 & 0 & 14655.2 & -1831.9 \\
0 & 0 & 0 & 0 & 0 & 0 & 1831.9 & -229.0 & 0 & -24000.0 & -1831.9 & 24229.0
\end{array}\right]
\begin{array}{c} 1 \\ \\ 2 \\ \\ 3 \\ \\ 4 \\ \\ 5 \\ \\ 6 \\ \\ \end{array}
\end{array}
$$

The global stiffness matrix after taking into consideration boundary conditions:

$$
\mathbf{K}^{r}=
\begin{array}{cc}
\begin{array}{cccccccccccc}
\multicolumn{2}{c}{1} & \multicolumn{2}{c}{2} & \multicolumn{2}{c}{3} & \multicolumn{2}{c}{4} & \multicolumn{2}{c}{5} & \multicolumn{2}{c}{6}
\end{array} & \\
\left[\begin{array}{cc|cc|cc|cc|cc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 19710.1 & 3012.2 & 0 & 0 & -1442.0 & 1321.9 & -9051.5 & 1810.3 & 0 & 0 \\
0 & 0 & 3012.2 & 10217.4 & 0 & -4547.4 & 1321.9 & -1211.7 & 1810.3 & -362.1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 19540.2 & -2442.5 & -9770.1 & 1221.3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -4547.4 & -2442.5 & 4852.7 & 1221.3 & -152.7 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -1442.0 & 1321.9 & -9770.1 & 1221.3 & 28169.0 & -2361.1 & -2301.7 & -2014.0 & -14655.2 & 0 \\
0 & 0 & 1321.9 & -1211.7 & 1221.3 & -152.7 & -2361.1 & 3355.6 & -2014.0 & -1762.2 & 1831.9 & 0 \\
\hline
0 & 0 & -9051.5 & 1810.3 & 0 & 0 & -2301.7 & -2014.0 & 11353.2 & 203.7 & 0 & 0 \\
0 & 0 & 1810.3 & -362.1 & 0 & 0 & -2014.0 & -1762.2 & 203.7 & 26124.3 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -14655.2 & 1831.9 & 0 & 0 & 14655.2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
&
\begin{array}{c}
1 \\ \\ 2 \\ \\ 3 \\ \\ 4 \\ \\ 5 \\ \\ 6 \\ \\
\end{array}
\end{array}
$$

The boundary conditions are described by the equations:

$u_{1X} = 0$, $u_{1Y} = 0$, $u_{6Y} = 0$.

Global numbers of the degrees of freedom of these displacements are equal to

$u_{1X} \to$ No 1; $u_{1Y} \to$ No 2; $u_{6Y} \to$ No 12, respectively.

We lead in the modification in the equations with the above numbers consisting in the insertion of zeros into rows of the matrix **K** and leading in 1 on the main diagonal of this matrix. After the symmetrization of the matrix (the insertion of zeros into suitable columns) we obtain the matrix $\mathbf{K}^r$ presented on the next page. The modified components are marked in italic fonts. As it was noticed in Chapter II the matrix $\mathbf{K}^r$ is a positively determined matrix, thus, its determinant has to be bigger than zero. For our matrix we calculate

$$\det(\mathbf{K}^r) = 217728.3382 \cdot 10^{30} \quad > 0\,.$$

Now we solve the set of equations

$$\mathbf{K}^r\,\mathbf{u} = \mathbf{p}^r\,,$$

with the use of the algorithm given in Appendix 2 we obtain the nodal displacement vector of the truss **u** and after inserting it into equation (2.75), we get the constraint reaction vector **r**:

$$\mathbf{u}=\begin{array}{cc}\left[\begin{array}{c} 0.0\\ 0.0\\ \hline -8.70282\text{E-}4\\ 6.01511\text{E-}4\\ \hline -1.94920\text{E-}4\\ 6.01511\text{E-}4\\ \hline -5.62319\text{E-}4\\ -1.76807\text{E-}4\\ \hline -1.24382\text{E-}3\\ 2.30635\text{E-}5\\ \hline -5.40219\text{E-}4\\ 0.0 \end{array}\right] & \begin{array}{c} 1\\ \\ 2\\ \\ 3\\ \\ 4\\ \\ 5\\ \\ 6\\ \\ \end{array}\end{array} \qquad \mathbf{r}=\begin{array}{cc}\left[\begin{array}{c} 6.964\\ 2.554\\ \hline \text{------}\\ \text{------}\\ \hline \text{------}\\ \text{------}\\ \hline \text{------}\\ \text{------}\\ \hline \text{------}\\ \text{------}\\ \hline \text{------}\\ -0.554 \end{array}\right] & \begin{array}{c} 1\\ \\ 2\\ \\ 3\\ \\ 4\\ \\ 5\\ \\ 6\\ \\ \end{array}\end{array}$$

The nodal displacement vector **u** can be used to draw the scheme of the deformation of the structure which is shown in Fig.2.E1.2.

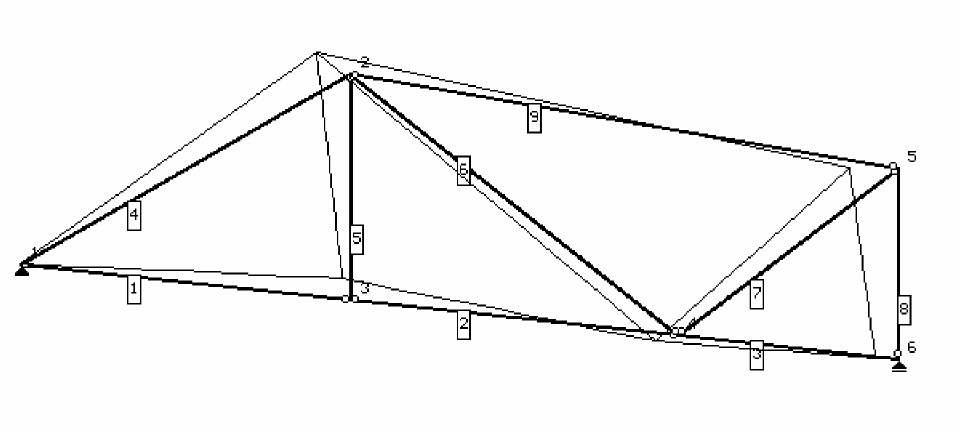

Fig.2.E1.2

On the basis of equation (2.76) we calculate internal forces for the elements and from equation (2.77) we determine stresses. The values of the nodal displacements, internal forces and stresses for the elements are given in Tab.2.E1.3.

Tab.2.E1.3

We obtain columns marked by u_{iX}, u_{iY}, u_{jX}, u_{jY} in the way shown in Fig.2.14

No n	u_{iX} [m]	u_{iY} [m]	u_{jX} [m]	u_{jY} [m]	u_{ix} [m] (2.22)	u_{jx} [m] (2.22)	N [kN] (2.76)	σ [kN/m^2] (2.77)
1	0.0	0.0	-1.94920E-4	6.01511E-4	0.0	-2.68023E-3	-2.660	-1063.8
2	-1.94920E-4	6.01511E-4	-5.62319E-4	-1.76807E-4	-2.68023E-3	-5.36046E-3	-2.660	-1063.8
3	-5.62319E-4	-1.76807E-4	-5.40219E-4	0.0	-5.36048E-3	-5.36049E-3	0.0	0.0
4	0.0	0.0	-8.70282E-4	6.01511E-4	0.0	-3.90460E-3	-5.198	-1299.5
5	-1.94920E-4	6.01511E-4	-8.70282E-4	6.01511E-4	6.01511E-3	6.01511E-3	0.0	0.0
6	-8.70282E-4	6.01511E-4	-5.62319E-4	-1.76807E-4	-1.04799E-2	-2.95043E-3	1.998	2220.1
7	-5.62319E-4	-1.76807E-4	-1.24382E-3	2.30635E-5	-5.39616E-3	-9.20881E-3	-1.549	-1721.6
8	-5.40219E-4	0.0	-1.24382E-3	2.30635E-5	0.0	2.30635E-4	0.554	138.4
9	-8.70282E-4	6.01511E-4	-1.24382E-3	2.30635E-5	-9.71348E-3	-1.22419E-2	-2.380	-595.0