CHAPTER V.
STATICS OF A 3D FRAME SYSTEM

A three dimensional frame structure is the most general type of bar structures. Elements of a space frame can serve for modelling of all previously described structures (2D and 3D trusses, 2D frames) and some others as grillworks, beams broken in a plane and loaded perpendicularly to its plane, etc. A few examples of structures which cannot be modelled by presented so far elements but can only be modelled with the help of 3D frame elements are presented in Fig.5.1.

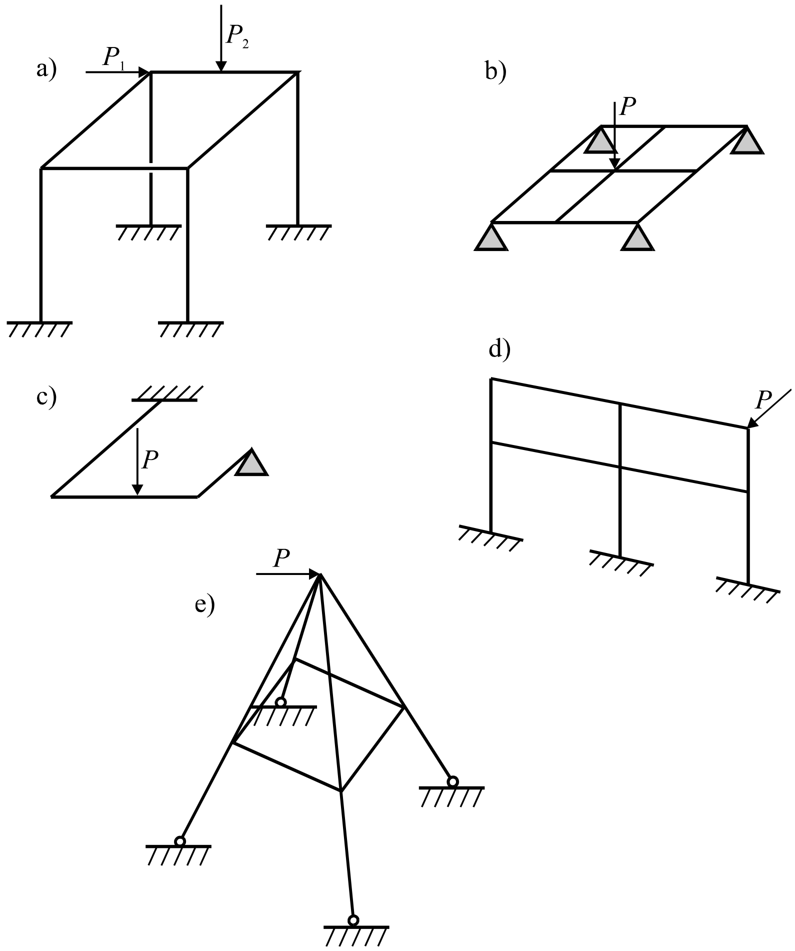
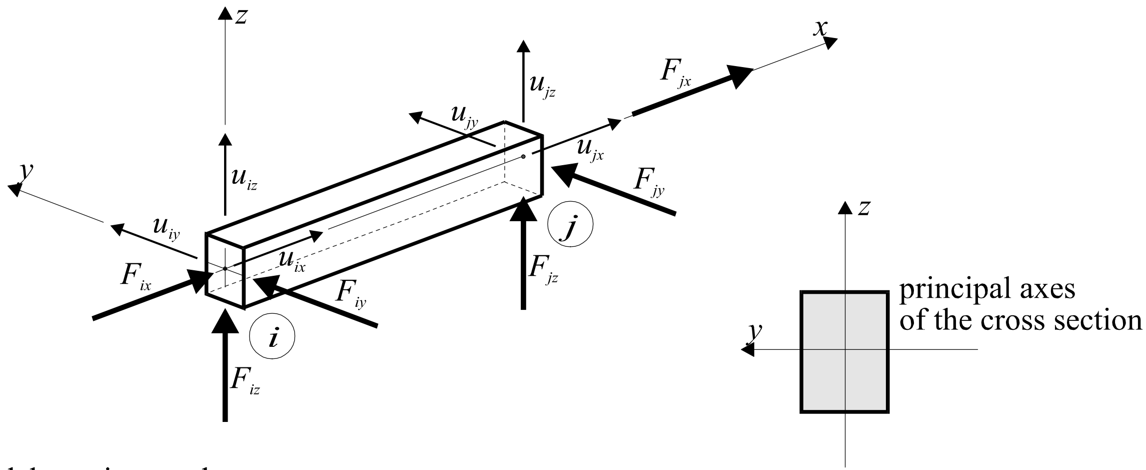


Fig.5.1

5.1. THE ELEMENT STIFFNESS MATRIX OF A 3D FRAME

Any node of a space structure has six degrees of freedom which means that it can submit to three independent displacements and three rotations. Hence a frame element has twelve degrees of freedom. Components of both nodal forces and displacements of the frame element are shown in Fig.5.2. The local coordinate system has to be chosen in such a way that axes y and z are the principal axes of a cross section because it simplifies the discussion of a bending problem. Bending such an element can be analysed as two independent phenomena of bending in planes xy and xz.

a) nodal displacements and forces



b) nodal rotations and moments

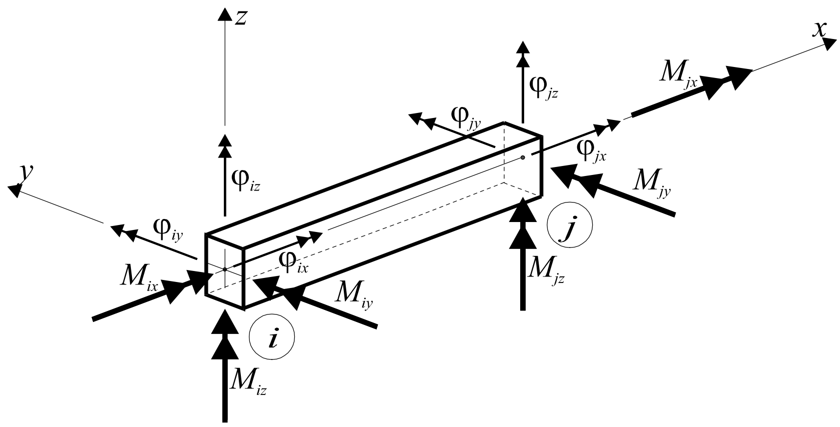


Fig.5.2

Here we will present nodal displacements and forces similarly, that is, in the form of vectors (column matrices).

The nodal displacement vector of an element in the local system

$$\mathbf{u'}^e = \begin{bmatrix} \mathbf{u'}_i \\ \mathbf{u'}_j \end{bmatrix}, \tag{5.1}$$

where

$$\mathbf{u'}_i = \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{iz} \\ \varphi_{ix} \\ \varphi_{iy} \\ \varphi_{iz} \end{bmatrix} - \tag{5.2}$$

is the displacement vector of the node i in the local coordinate system,

$$\mathbf{u'}_j = \begin{bmatrix} u_{jx} \\ u_{jy} \\ u_{jz} \\ \varphi_{jx} \\ \varphi_{jy} \\ \varphi_{jz} \end{bmatrix} - \tag{5.3}$$

is the displacement vector of the node j in the local coordinate system.

The nodal force vector of an element in the local system

$$\mathbf{f'}^e = \begin{bmatrix} \mathbf{f'}_i \\ \mathbf{f'}_j \end{bmatrix}, \tag{5.4}$$

where

$$\mathbf{f'}_i = \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \end{bmatrix} - \tag{5.5}$$

is the force vector of the node i in the local coordinate system,

$$\mathbf{f'}_j = \begin{bmatrix} F_{jx} \\ F_{jy} \\ F_{jz} \\ M_{jx} \\ M_{jy} \\ M_{jz} \end{bmatrix} - \tag{5.6}$$

is the force vector of the node j in the local coordinate system.

As usual we look for the relation between nodal forces and displacements in the form:

$$\mathbf{f'}^e = \mathbf{K'}^e\, \mathbf{u'}^e, \tag{5.7}$$

where the stiffness matrix $\mathbf{K}'^e$ is a quadric and symmetric matrix with dimensions 12x12. Most components of this matrix can be calculated on the basis of the results obtained for a 2D frame in Chapter IV. Since the phenomena of bending in principal planes of the cross section do not influence each other, we will split the deformation of the element of a three-dimensional frame into a few simpler forms:

- axial tension which is identical to that in a truss,
- bending in the plane *xz* which is similar to the states of a 2D frame; modifications concern signs of internal forces,
- torsion.

As it is seen, torsion of a frame element is a state which has not been described so far. The dependence between a nodal torsion moment and a torsion angle of an element is quite simple (comp. [8]) and resembles the relation between an axial force and an element extension:

$$\frac{\Delta\varphi_x}{L} = \frac{M_x}{GC}, \tag{5.8}$$

where $\Delta\varphi_x = \varphi_{jx} - \varphi_{ix}$ is the increase in the torsion angle due to the torsion moment M_x ,

$G = \dfrac{E}{2(1+\nu)}$ - is Kirchhoff's modulus and C is the torsional resistance.

The constant C has the dimension of a moment of inertia and is equal to the polar moment of inertia for circular-symmetric sections (comp. [8]) but for other sections it should be calculated by use of quite complex methods (comp. [17]). The calculation method of this constant for a few often occurring cross sections in engineering practice are given in Appendix 3.

Equation (5.8) allows to write the relation between the nodal rotations around the x axis and nodal torsion moments:

$$\begin{aligned} M_{ix} &= \frac{GC}{L}\left(\varphi_{ix} - \varphi_{jx}\right), \\ M_{jx} &= \frac{GC}{L}\left(-\varphi_{ix} + \varphi_{jx}\right). \end{aligned} \tag{5.9}$$

The above equations are the searched relation which allows to write the element stiffness matrix. Senses of nodal forces caused by unitary nodal displacements which allow to determine signs of the expressions of the stiffness matrix are shown in Fig.5.3.

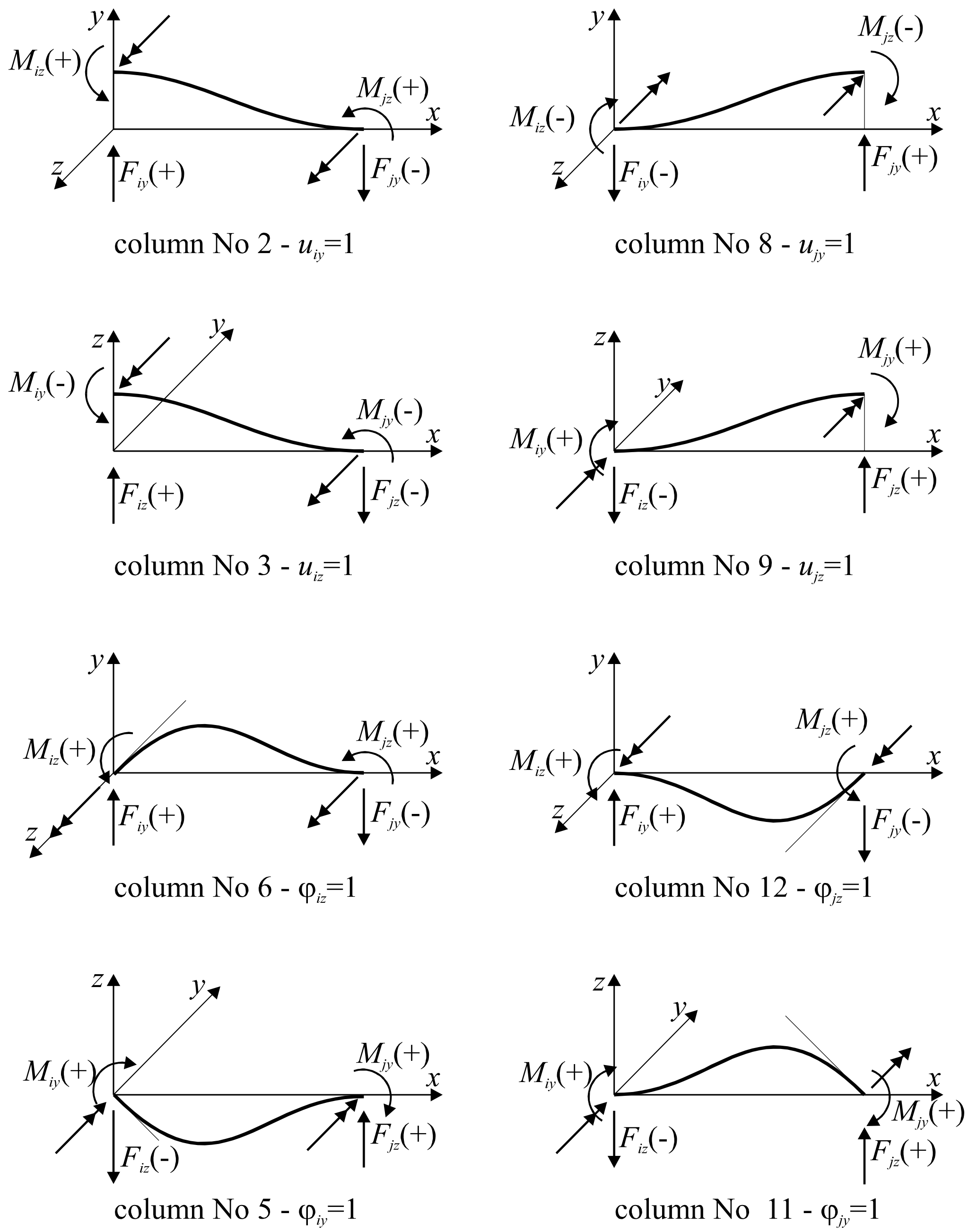


Fig.5.3. The signs of the nodal force vector caused by unitary nodal displacements.

The element stiffness matrix is presented by equation

$$
\mathbf{K'}^{e} = \begin{array}{c}
\begin{array}{cccccccccccc} u_{ix} & u_{iy} & u_{iz} & \varphi_{ix} & \varphi_{iy} & \varphi_{iz} & u_{jx} & u_{jy} & u_{jz} & \varphi_{jx} & \varphi_{jy} & \varphi_{jz} \end{array} \\
\left[\begin{array}{cccccc|cccccc}
\frac{EA}{L} & & & & & & -\frac{EA}{L} & & & & & \\
 & 12\frac{EJ_z}{L^3} & & & & 6\frac{EJ_z}{L^2} & & -12\frac{EJ_z}{L^3} & & & & 6\frac{EJ_z}{L^2} \\
 & & 12\frac{EJ_y}{L^3} & & -6\frac{EJ_y}{L^2} & & & & -12\frac{EJ_y}{L^3} & & -6\frac{EJ_y}{L^2} & \\
 & & & \frac{GC}{L} & & & & & & -\frac{GC}{L} & & \\
 & & -6\frac{EJ_y}{L^2} & & 4\frac{EJ_y}{L^2} & & & & 6\frac{EJ_y}{L^2} & & 2\frac{EJ_y}{L^2} & \\
 & 6\frac{EJ_z}{L^2} & & & & 4\frac{EJ_z}{L^2} & & -6\frac{EJ_z}{L^2} & & & & 2\frac{EJ_z}{L^2} \\
\hline
-\frac{EA}{L} & & & & & & \frac{EA}{L} & & & & & \\
 & -12\frac{EJ_z}{L^3} & & & & -6\frac{EJ_z}{L^2} & & 12\frac{EJ_z}{L^3} & & & & -6\frac{EJ_z}{L^2} \\
 & & -12\frac{EJ_y}{L^3} & & 6\frac{EJ_y}{L^2} & & & & 12\frac{EJ_y}{L^3} & & 6\frac{EJ_y}{L^2} & \\
 & & & -\frac{GC}{L} & & & & & & \frac{GC}{L} & & \\
 & & -6\frac{EJ_y}{L^2} & & 2\frac{EJ_y}{L^2} & & & & 6\frac{EJ_y}{L^2} & & 4\frac{EJ_y}{L^2} & \\
 & 6\frac{EJ_z}{L^2} & & & & 2\frac{EJ_z}{L^2} & & -6\frac{EJ_z}{L^2} & & & & 4\frac{EJ_z}{L^2}
\end{array}\right]
\end{array}
\begin{array}{c} \\ F_{ix} \\ F_{iy} \\ F_{iz} \\ M_{ix} \\ M_{iy} \\ M_{iz} \\ F_{jx} \\ F_{jy} \\ F_{jz} \\ M_{jx} \\ M_{jy} \\ M_{jz} \end{array}
\qquad (5.10)
$$

5.2. TRANFORMATION OF THE STIFFNESS MATRIX TO THE GLOBAL COORDINATE SYSTEM

The element stiffness matrix should be transformed to the global system. The transformation method of the matrix of a frame element is analogous to the transformation of an element of a 3D truss presented in Chapter III but the third rotation around the x axis of the local system is necessary in order to lead axes y and z to the position of the principal central axes of inertia of an element cross section. Such a choice of local axes is very important for building the stiffness matrix which has been noted at the beginning of this chapter. The location of an element in space, applied types of coordinate systems and rotation angles notations are presented in Fig.5.4.

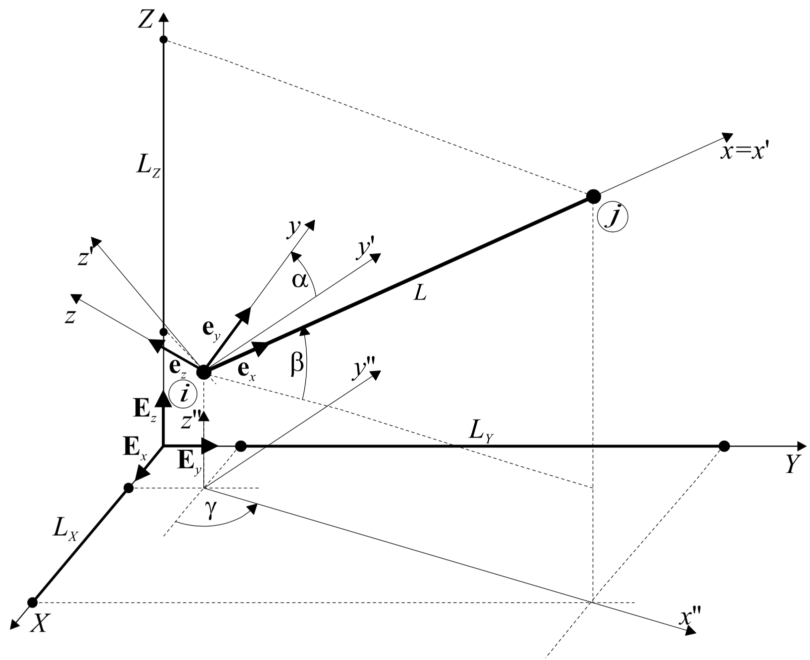


Fig.5.4

In Fig.5.4 the following notations are led: $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ as basic vectors of axes of the local coordinate system and $\mathbf{E}_X$, $\mathbf{E}_Y$, $\mathbf{E}_Z$ as basic vectors of axes of the global coordinate system. They will be helpful in subsequent transformations.

5.2.1. Use of the rotation angle α for building the transformation matrix

Now we perform the transformation of a certain displacement vector $\mathbf{u'}_i$ from the local system to the global one by the composition of three rotations:

$$\mathbf{u}_i = \mathbf{R}_\gamma \left[\mathbf{R}_\beta \left(\mathbf{R}_\alpha \mathbf{u}'_i \right) \right], \tag{5.11}$$

where $$\mathbf{R}_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha \\ 0 & s_\alpha & c_\alpha \end{bmatrix}, \tag{5.12}$$

is the rotation matrix around the x axis by an angle α,

$$\mathbf{R}_\beta = \begin{bmatrix} c_\beta & 0 & -s_\beta \\ 0 & 1 & 0 \\ s_\beta & 0 & c_\beta \end{bmatrix}, \tag{5.13}$$

is the rotation matrix around the y' axis by an angle β,

$$\mathbf{R}_\gamma = \begin{bmatrix} c_\gamma & -s_\gamma & 0 \\ s_\gamma & c_\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5.14}$$

is the rotation matrix around the z'' axis by an angle γ. In equations (5.12), (5.13) and (5.14) we have $c_\alpha = \cos\alpha$, $s_\alpha = \sin\alpha$, $c_\beta = \cos\beta$, $s_\beta = \sin\beta$, $c_\gamma = \cos\gamma$ and $s_\gamma = \sin\gamma$. Equation (5.11) can be written in a simpler way:

$$\mathbf{u}_i = \mathbf{R}_i \mathbf{u}'_i , \tag{5.15}$$

where $\mathbf{R}_i = \mathbf{R}_\gamma \mathbf{R}_\beta \mathbf{R}_\alpha$ is the transformation matrix and the inverse relation is:

$$\mathbf{u}'_i = \left(\mathbf{R}_i\right)^{\mathrm{T}} \mathbf{u}_i , \tag{5.16}$$

where $\left(\mathbf{R}_i\right)^{\mathrm{T}} = \left(\mathbf{R}_\alpha\right)^{\mathrm{T}} \left(\mathbf{R}_\beta\right)^{\mathrm{T}} \left(\mathbf{R}_\gamma\right)^{\mathrm{T}}$.

With this way of transformation functions of angles γ and β can be determined on the basis of nodal coordinates of an element (they depend on direction cosines of an element - comp. Sec.3.2) and the angle α is an additional parameter which has to be given for all elements.

5.2.2. Use of a direction vector

At this moment we will present another way of determining the transformation matrix. Let an additional parameter determining an element be a direction vector $\mathbf{e}_y$ (Fig.5.4) which is located on the y axis of the local system and its module is equal to a unite (such a vector is called a basic vector or a versor of an axis). Hence we have

- vector of the x axis of the local system determined on the basis of element coordinates (its components are direction cosines of the element)

$$\mathbf{e}_x = \begin{bmatrix} e_{xX} \\ e_{xY} \\ e_{xZ} \end{bmatrix} = \frac{1}{L}\begin{bmatrix} L_X \\ L_Y \\ L_Z \end{bmatrix}, \tag{5.17}$$

- given direction vector of the element

$$\mathbf{e}_y = \begin{bmatrix} e_{yX} \\ e_{yY} \\ e_{yZ} \end{bmatrix}. \tag{5.18}$$

We look for the third basic vector $\mathbf{e}_z$ which allows to write the transformation of any vector from the local coordinate system xyz to the global one XYZ.

Since the system xyz is the right cartesian coordinate system, then the versors of this system are orthogonal. Thus, we can write

$$\mathbf{e}_z = \mathbf{e}_x \times \mathbf{e}_y, \tag{5.19}$$

and from here we calculate

$$\mathbf{e}_z = \begin{bmatrix} e_{zX} \\ e_{zY} \\ e_{zZ} \end{bmatrix}, \tag{5.20}$$

where

$$e_{zX} = \begin{vmatrix} e_{xY} & e_{xZ} \\ e_{yY} & e_{yZ} \end{vmatrix},\ e_{zY} = -\begin{vmatrix} e_{xX} & e_{xZ} \\ e_{yX} & e_{yZ} \end{vmatrix},\ e_{zZ} = \begin{vmatrix} e_{xX} & e_{xY} \\ e_{yX} & e_{yY} \end{vmatrix}, \tag{5.21}$$

are the coordinates of the versor of the local z axis with regard to the global coordinate system.

Since any vector can be presented as a sum of products of its coordinates and versors, then we obtain:

$$\begin{aligned} \mathbf{u} &= u_x\mathbf{e}_x + u_y\mathbf{e}_y + u_z\mathbf{e}_z = u_x\left(e_{xX}\mathbf{E}_X + e_{xY}\mathbf{E}_Y + e_{xZ}\mathbf{E}_Z\right) + \\ &+ u_y\left(e_{yX}\mathbf{E}_X + e_{yY}\mathbf{E}_Y + e_{yZ}\mathbf{E}_Z\right) + u_z\left(e_{zX}\mathbf{E}_X + e_{zY}\mathbf{E}_Y + e_{zZ}\mathbf{E}_Z\right) = \\ &= \left(u_x e_{xX} + u_y e_{yX} + u_z e_{zX}\right)\mathbf{E}_X + \left(u_x e_{xY} + u_y e_{yY} + u_z e_{zY}\right)\mathbf{E}_Y + \\ &+ \left(u_x e_{xZ} + u_y e_{yZ} + u_z e_{zZ}\right)\mathbf{E}_Z, \end{aligned} \tag{5.22}$$

or less

$$\mathbf{u}_i = \mathbf{R}_i\mathbf{u}'_i, \tag{5.23}$$

where $\mathbf{R}_i$ is the rotation matrix of a node

$$\mathbf{R}_i = \begin{bmatrix} e_{xX} & e_{yX} & e_{zX} \\ e_{xY} & e_{yY} & e_{zY} \\ e_{xZ} & e_{yZ} & e_{zZ} \end{bmatrix}. \tag{5.24}$$

5.2.3. Use of a direction point

The necessity to give the direction vector in form (5.18) often causes difficulties during the input of data. Here we present one of the possibilities of simplifying the way of passing the direction of an element axis which is used in the ALGOR system. The 3D frame element is determined by three points (i - the first node, j - the last node, k - the direction node) in it. The points i, j, k determine a plane in the three dimensional space. The axis y of the local coordinate system is in this plane. The x axis is determined by the line passing through points i, j. We find coordinates of versors for such a definition of directions of the local axes. Let X_i, Y_i, Z_i denote coordinates of the point i in the global system. If by analogy to them we denote coordinates of points j and k, then the element coordinates in the global system are equal to

$$L_X = X_j - X_i,\ L_Y = Y_j - Y_i,\ L_Z = Z_j - Z_i,\ L = \sqrt{L_X^2 + L_Y^2 + L_Z^2}\ , \tag{5.25}$$

and from here we calculate the components of vector $\mathbf{e}_x$:

$$e_{xX} = \frac{L_X}{L},\ e_{xY} = \frac{L_Y}{L},\ e_{xZ} = \frac{L_Z}{L}. \tag{5.26}$$

We form the vector $\mathbf{v}$ connecting the point i and the direction point k (Fig.5.5):

$$\mathbf{v} = \begin{bmatrix} X_k - X_i \\ Y_k - Y_i \\ Z_k - Z_i \end{bmatrix}. \tag{5.27}$$

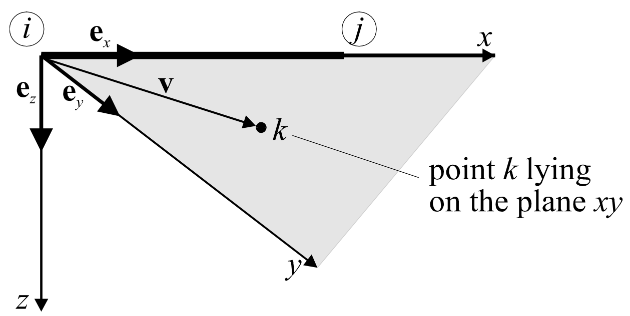


Fig.5.5

The vector product of the vectors $\mathbf{e}_x$ and $\mathbf{v}$ give a vector which is perpendicular to the plane xy. This vector will be the versor $\mathbf{e}_z$:

$$\mathbf{w} = \mathbf{e}_x \times \mathbf{v}, \tag{5.28}$$

$$w_X = \begin{vmatrix} e_{xY} & e_{xZ} \\ v_Y & v_Z \end{vmatrix},\ w_Y = -\begin{vmatrix} e_{xX} & e_{xZ} \\ v_X & v_Z \end{vmatrix},\ w_Z = \begin{vmatrix} e_{xX} & e_{xY} \\ v_X & v_Y \end{vmatrix}$$

$$w = \sqrt{w_X^2 + w_Y^2 + w_Z^2}, \tag{5.29}$$

$$\mathbf{e}_{zX} = \frac{w_X}{w},\ \mathbf{e}_{zY} = \frac{w_Y}{w},\ \mathbf{e}_{zZ} = \frac{w_Z}{w}. \tag{5.30}$$

Now we obtain the coordinates of the versor $\mathbf{e}_y$ from the vector product of the versor $\mathbf{e}_z$ by $\mathbf{e}_x$:

$$\mathbf{e}_y = \mathbf{e}_z \times \mathbf{e}_x, \tag{5.31}$$

$$e_{yX} = \begin{vmatrix} e_{zY} & e_{zZ} \\ e_{xY} & e_{xZ} \end{vmatrix},\ e_{yY} = -\begin{vmatrix} e_{zX} & e_{zZ} \\ e_{xX} & e_{xZ} \end{vmatrix},\ e_{yZ} = \begin{vmatrix} e_{zX} & e_{zY} \\ e_{xX} & e_{xY} \end{vmatrix}. \tag{5.32}$$

On the basis of results (5.26), (5.30) and (5.32) we can form the transformation matrix $\mathbf{R}_i$ as in equation (5.24).

5.2.4. The transformation matrix of an element

Now we build the transformation matrix of an element. Nodal displacement vectors and nodal force vectors have been grouped so that we can divide them into blocks containing either displacements or rotations and either forces or moments respectively. After this operation we can transform every block independently

$$\mathbf{R}^e = \begin{bmatrix} \mathbf{R}_i & & & \\ & \mathbf{R}_i & & \\ & & \mathbf{R}_j & \\ & & & \mathbf{R}_j \end{bmatrix}, \tag{5.33}$$

where $\mathbf{R}_i$ is the rotation matrix of the node i and $\mathbf{R}_j$ is the rotation matrix of the node j. Since the element is straight, so as it was in previous cases (2D and 3D trusses, 2D frame), we assume $\mathbf{R}_i = \mathbf{R}_j$.

We obtain the transformation of the stiffness matrix to the global system by multiplying matrices identically as in equation **Błąd! Nie można odnaleźć źródła odwołania.**).

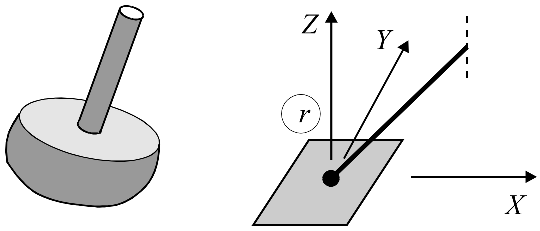
$$\mathbf{K}^e = \mathbf{R}^e \mathbf{K}'^e \left(\mathbf{R}^e\right)^{\mathrm{T}}, \tag{5. 34}$$

where $\mathbf{R}^e$ is determined by equation (5.33). The form of the matrix $\mathbf{K}^e$ is too complex in the global system, so we will not give it.

5.3. BOUNDARY CONDITIONS FOR A 3D FRAME

Boundary conditions existing in 3D frame supports are very similar to conditions described for two-dimensional frames. Differences concerning degrees of freedom which do not exist in plate frames are obvious. We elaborate only those boundary conditions which describe frame supports of space structures (Fig.5.6) and which are most often applied.

a) rigid support

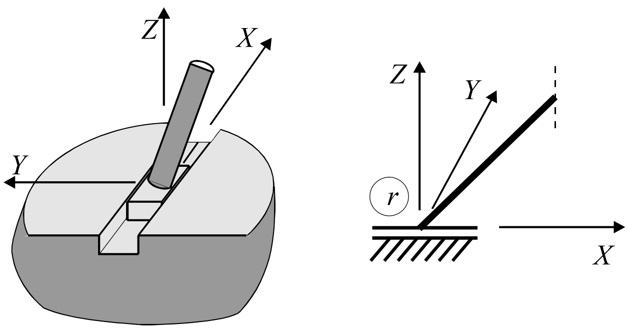


$u_{rX} = 0 \quad \varphi_{rX} = 0$

$u_{rY} = 0 \quad \varphi_{rY} = 0$

$u_{rZ} = 0 \quad \varphi_{rZ} = 0$

b) linear moveable support (along the axis *X*)

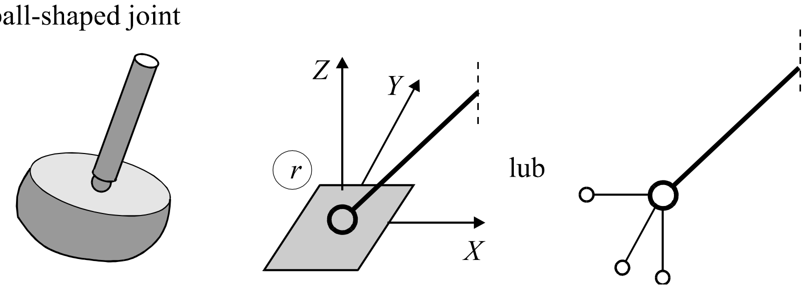


$u_{rY} = 0 \quad \varphi_{rX} = 0$

$u_{rZ} = 0 \quad \varphi_{rY} = 0$

$\varphi_{rZ} = 0$

c) ball-shaped joint

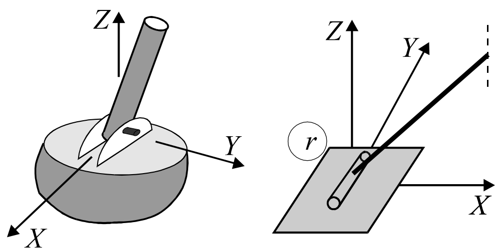


$u_{rX} = 0$

$u_{rY} = 0$

$u_{rZ} = 0$

d) cylindrical joint (rotation around the axis *Y*)

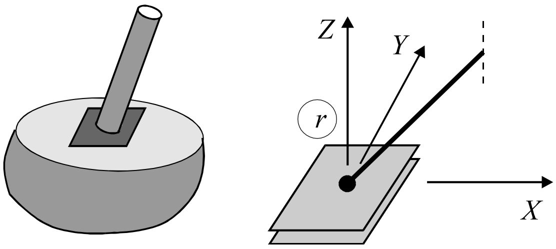


$u_{rX} = 0 \quad \varphi_{rX} = 0$

$u_{rY} = 0 \quad \varphi_{rZ} = 0$

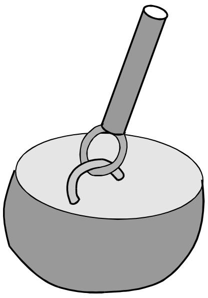
$u_{rZ} = 0$

e) moveable plane support (a displacement on the plane *XY*)



$u_{rZ} = 0$

f) Cardan joint



$u_{rX} = 0 \quad \varphi_{rZ} = 0$

$u_{rY} = 0$

$u_{rZ} = 0$

Fig.5.6

Modification of the global stiffness matrix (comp. point 2.6) is the way of considering boundary conditions just as we have done it in reference to previously described structures.

5.4. BOUNDARY ELEMENTS

A choice of supports possible to be used in a space structure increases if we add elastic constraints and „skew" supports.

As in previous chapters we propose to use elastic and fixed boundary elements for modelling these constraints. In fact we can use a single element described in Chapters II or III of which we can compose a more complex support but for convenience we will show here the use of the matrix of a versatile elastic element with six degrees of freedom:

$$\mathbf{K}'^{b} = \begin{bmatrix} h_{rX} & 0 & 0 & 0 & 0 & 0 \\ 0 & h_{rY} & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{rZ} & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{rX} & 0 & 0 \\ 0 & 0 & 0 & 0 & g_{rY} & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{rZ} \end{bmatrix}, \tag{5.35}$$

where h_{rX}, h_{rY}, h_{rZ} are spring rates and g_{rX}, g_{rY}, g_{rZ} are flexural (or torsion) stiffness of springs.

The transformation of this matrix to the global system is similar to the one presented in Chapter IV (equation**Błąd! Nie można odnaleźć źródła odwołania.**)). Since reactions of our elements are contained in two independent vectors: the vector of support forces and the vector of support moments, then the transformation matrix has the form:

$$\mathbf{R}^b = \begin{bmatrix} \mathbf{R}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_r \end{bmatrix}, \tag{5.36}$$

where $\mathbf{R}_r$ is the rotation matrix of the node given by equation (5.24). After the multiplication we obtain the stiffness matrix of the boundary element in the global coordinate system:

$$\mathbf{K}^b = \mathbf{R}^b \mathbf{K}'^b \left(\mathbf{R}^b\right)^{\mathrm{T}} = \begin{bmatrix} \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} \end{bmatrix}, \tag{5.37}$$

where **H** is the stiffness matrix for a movement and **G** is the stiffness matrix for a rotation:

$$\mathbf{H} = \begin{bmatrix} e_{xX}^2 h_{rX} + e_{yX}^2 h_{rY} + e_{zX}^2 h_{rZ} & 0 & 0 \\ 0 & e_{xY}^2 h_{rX} + e_{yY}^2 h_{rY} + e_{zY}^2 h_{rZ} & 0 \\ 0 & 0 & e_{xZ}^2 h_{rX} + e_{yZ}^2 h_{rY} + e_{zZ}^2 h_{rZ} \end{bmatrix}. \tag{5.38}$$

It is easy to obtain the matrix **G** from the matrix **H** changing the stiffness of tension of springs h_{rX}, h_{rY}, h_{rZ} into the stiffness of bending springs g_{rX}, g_{rY}, g_{rZ}.