APPENDIX 3
STIFFNESS OF TORSION FRAME ELEMENTS

The problem of torsion bars is very important in practice. The determination of the bar stiffness in the process of torsion is necessary to determine components of stiffness indices of 3D frame elements (comp. Chapter V). The problem of determination of stress and stiffness of a bar with a circular symmetric cross section (Fig.A3.1) was solved by Coulomb at the end of 18^{th} century [17].

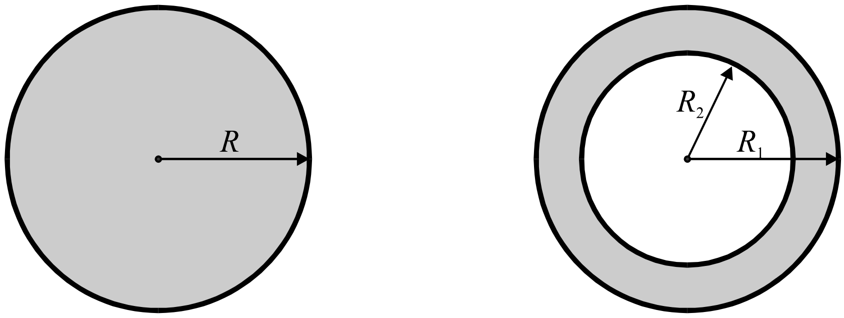

Fig.A3.1

A. A circular cross section

In case of circular cross sections their torsion stiffness is equal to the polar moment of inertia and:

$$C = J_o = \frac{\pi R^4}{2} \text{ for a full circular cross section}$$

and

$$C = J_o = \frac{\pi}{2}\left(R_1^4 - R_2^4\right) \text{ for a pipe cross section.}$$

Thus, the dependence between the torsion moment M_s and a unit angle of a cross section rotation is equal to

$M_s = CG\vartheta$.

The problem of determination of stiffness and stress in a torsion bar with any cross section was solved by de Saint-Venant in the middle of 19^{th} century. He assumed that non-circular cross sections undergo deplanation. The determination of a warping function requires solving a harmonic differential equation:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0\,.$$

Many ways of solving this problem for different cross sections can be found in the book written by P.S.Timoshenko and J.N.Goodier [17] and another one written by M.T.Huber [6]. In this Appendix we give ready made solutions for a few different from the technical point of view cross sections.

B. An elliptic cross section

This problem was solved by de Saint-Venant in 1855.

$$C=\pi\frac{a^3b^3}{a^2+b^2}\,,$$

where *a* and *b* are half axes of an ellipse.

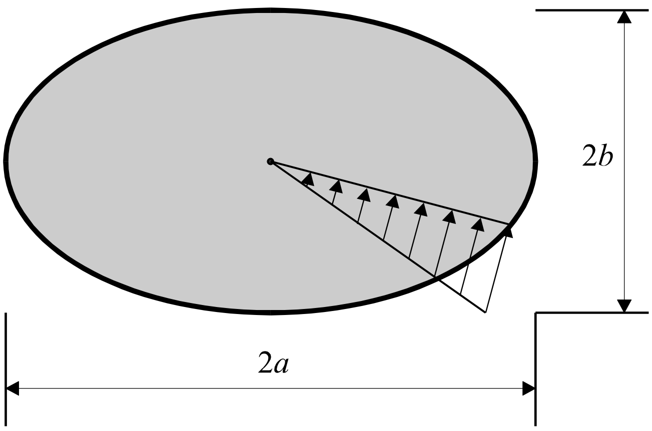

Fig.A3.2

C. An equilateral triangle

This problem was solved by de Saint-Venanta in 1855.

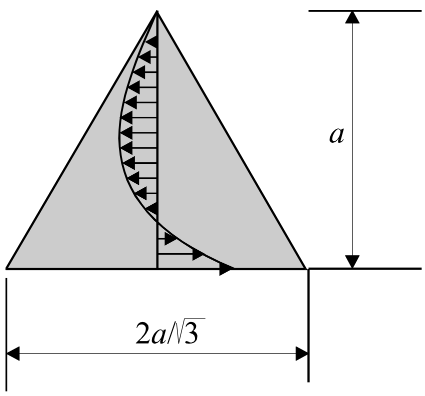

Fig.A3.3

$$C = \frac{a^4 \sqrt{3}}{45}$$

D. A rectangular cross section

That problem was solved by de Saint-Venanta in 1856.

$$\frac{a}{b} \le 1$$

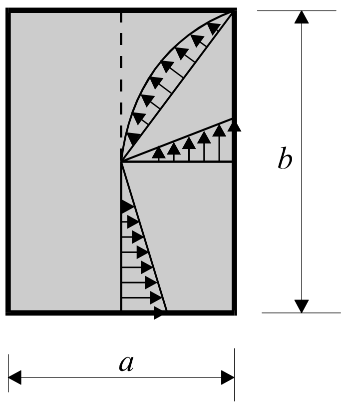

Fig.A3.4

$$C = k\left(\frac{a}{b}\right) a^3 b, \text{ where } k\left(\frac{a}{b}\right) = \frac{1}{3} - \frac{64}{\pi^5}\frac{a}{b}\sum_{n=1,3,5}^{\infty}\frac{1}{n^5}\,\mathrm{tgh}\,\frac{n\pi b}{2a}$$

Proper approximation can be obtained by using the formula:

$$k\left(\frac{a}{b}\right) \approx \frac{1}{3} - 0.21\frac{a}{b}\left(1 - \frac{a^4}{12b^4}\right),$$

giving the value which differs from the exact value not more than by 0.55% (at $\frac{a}{b} \approx 0.875$).

Fig.A3.5

The graph shows the dependence $k\left(\frac{a}{b}\right)$ which can be used for approximate determination of stiffness of a rectangular cross section (Fig.A3.5).

E. A circular segment

This problem was solved by de Saint-Venanta in 1878.

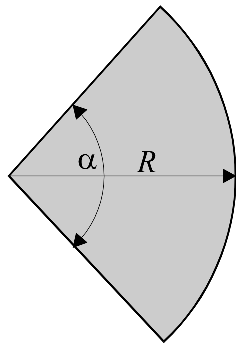

Fig.A3.6

$C = k(\alpha)R^4$,

where $k(\alpha)$ is the coefficient calculated on the basis of the equation:

$$k(\alpha) = \int_{-\alpha/2}^{\alpha/2}\int_{0}^{R}\left[-r^2\left(1-\frac{\cos 2\varphi}{\cos\alpha}\right)+\frac{16r^2\alpha^2}{\pi^3}\sum_{n=1.3.5}^{\infty}(-1)^{\frac{n+1}{2}}\left(\frac{r}{R}\right)^{\frac{n\pi}{\alpha}}\frac{\cos\frac{n\pi\varphi}{\alpha}}{n\left(n\frac{2\alpha}{\pi}\right)\left(n-\frac{2\alpha}{\pi}\right)}\right]rd\varphi dr\,.$$

We give the values of this coefficient for a few values of the angle α in the table below:

α	π/4	π/3	π/2	2π/3	π	3π/2	5π/3	2π
k	0.0181	0.0349	0.0825	0.148	0.296	0.572	0.672	0.878

F. An isosceles right-angled triangle

The above problem was solved by Galerkin in 1919.

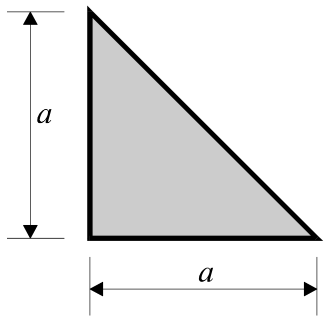

Fig.A3.7

$$C = \frac{a^4}{38.3}$$

G. A regular hexagon

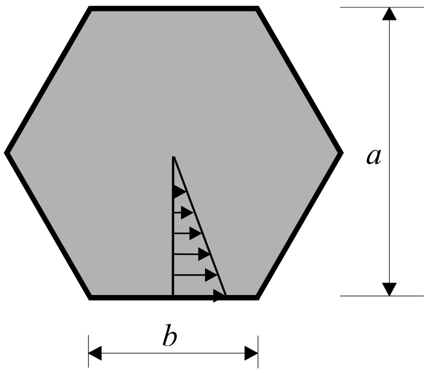

Fig.A3.8

$$J_o = \frac{5\sqrt{3}}{8} b^4 \;;\; A = \frac{3\sqrt{3}}{2} b^2 \;;$$

$$C = 1.0366\, b^4 \;;$$

$$\beta = \frac{A^4}{CJ_o} = 40.603 \;;\; \tau_{\max} = 0.564\, b^2 .$$

H. A thin-walled pipe with any cross section

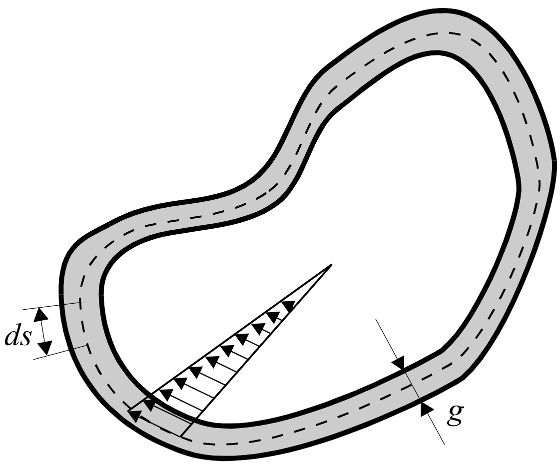

Fig.A3.9

$$C = \frac{4 A_o^2}{\int\limits_S \frac{ds}{g(\alpha)}} ,$$

where A_o is the surface of a figure limited by a line dividing the thickness of a pipe wall into halves. Integration should be done along the circuit S of this figure.

I. A thin-walled pipe cut along generating line

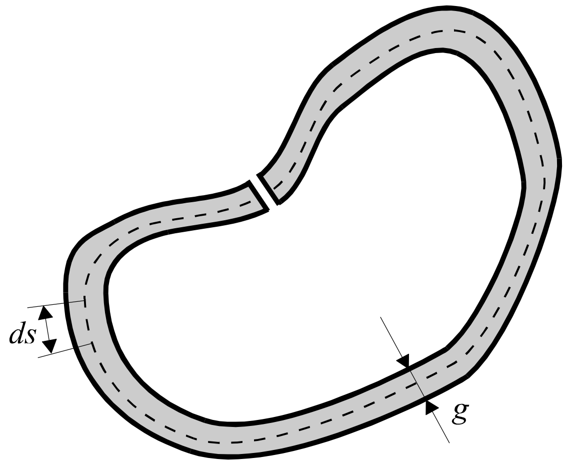

Fig.A3.10

$$C = \frac{1}{3}\int_S g^3 ds\,.$$

It is interesting to notice that stiffness does not depend on the shape of a cross section but it depends on its thickness and circuit *S*.

J. Cross sections composed of thin-walled rectangles

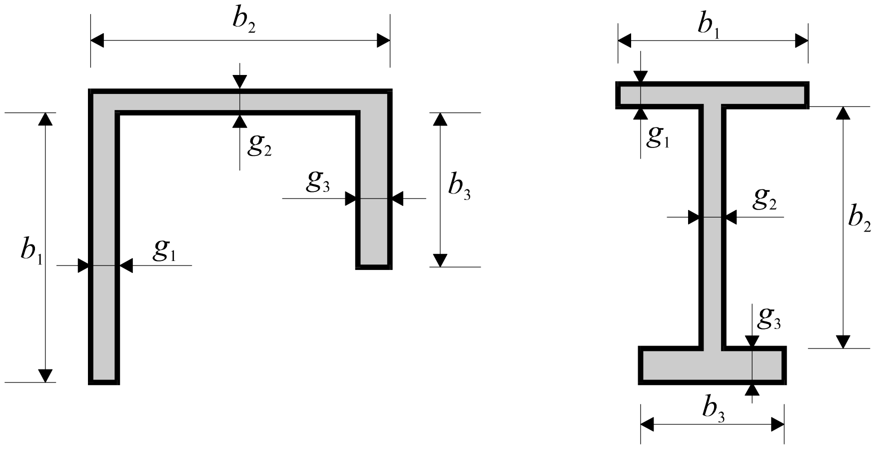

Fig.A3.11

$$C = \frac{1}{3}\sum_{i=1}^{n} g_i^3 b_i$$

Comparing coefficient $\frac{1}{3}$ in the above formula with the graph shown in Fig.A3.5, we note that stiffness is always overevaluated. For a cross section composed of rectangles with the same thickness more exact results are obtained by using the formula for rectangles (example D) where we substitute *g* for *a* and the length of a circuit of the middle line of a cross section is substituted for

$$b = \sum_{i=1}^{n} b_i \; .$$

K. A thick-walled pipe cut along generating line

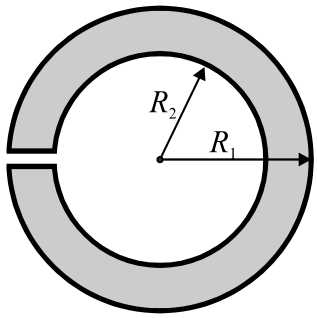

Fig.A3.12

$$C = \frac{\pi}{2}\left[R_2^4 - R_1^4 - \frac{\left(R_2^2 - R_1^2\right)^2}{\ln\left(\frac{R_1}{R_0}\right)} \right]$$

L. Other cross sections with crowned contour

On the basis of many exact solutions de Saint-Venant proposed to determine the torsion stiffness from the approximate formula:

$$C = \frac{A^4}{4\pi^2 J_o},$$

where A is the surface of a cross section and J_o is the center moment of inertia.

The above formula is exact for an ellipse. Generalising it, we write

$$C = \frac{A^4}{\beta J_o},$$

where β is the coefficient depending on the shape of a cross section. The table below in which you can find several different values of the coefficient β can be helpful as a reference.

Section	Circle, ellipse	Equilateral triangle	Rectangle				Circular segment		Isosceles right-angled triangle	Regular hexagon
			1:1	2:3	1:2	1:4	$\alpha=\pi/2$	$\alpha=\pi$		
Example	A, B	C	D	D	D	D	E	E	F	G
β	$4\pi^2=$ 39.478	45	42.674	42.438	41.976	40.221	42.022	40.935	43.088	40.603