CHAPTER III.
STATICS OF 3D TRUSS STRUCTURES

Although 3D truss structures have been known for a long time (comp. [17]), they have been used very rarely till now. It could be caused by much calculation trouble which an engineer designing a construction had to overcome. Though the series method simplifying the calculation of internal forces (the method of a nodal equilibrium and its graphic variant - Cremona's method and the method of sections - Ritter's method, etc.) has been devised for statically determined plate trusses, yet in case of space trusses only the method of a nodal equilibrium has remained. Large sets of equations which are generated by this method for space trusses have discouraged from designing this type of constructions. 3D structures looking as trusses, in fact, are seldom trusses, for instance, the famous Eiffel's tower or support columns of overhead power lines, masts (in particular with the quadrangular crosses) are most often space frames because they keep their geometric stability thanks to bended elements which do not exist in classical trusses. Both the use of computers and new methods of statics analysis of a structure making use of new technical possibilities (the finite element method is one of the main methods among them) have caused considerable progress in designing space trusses.

We think that one of the most popular use of these structures are structural roofs. Examples of space trusses are presented in Fig.3.1.

Fig.3.1

3.1. NOTATIONS AND BASIC RELATIONS

The node of a space truss has three degrees of freedom because while describing its movement we have to give three components of a displacement vector. The displacement vector and forces acting on an element of the space truss are shown in Fig.3.2. As in Chapter II components of forces and displacements vector are collected in column matrices which will be called vectors;

- nodal displacements vector of the first node *i* in the global coordinate system:

$$\mathbf{u}_i = \begin{bmatrix} u_{iX} \\ u_{iY} \\ u_{iZ} \end{bmatrix}, \tag{3.1}$$

- the same vector in the local coordinate system:

$$\mathbf{u'}_i = \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{iz} \end{bmatrix}, \tag{3.2}$$

- vector of nodal forces acting at the first node i of an element written in the global system:

$$\mathbf{f}_i = \begin{bmatrix} F_{iX} \\ F_{iY} \\ F_{iZ} \end{bmatrix}, \tag{3.3}$$

and in the local system:

$$\mathbf{f'}_i = \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{iz} \end{bmatrix}. \tag{3.4}$$

The above vectors form forces and displacements vectors of an element:

- vector of the nodal displacements of an element e with the node i (the first one) and j (the last one) is written in the global coordinate system as follows:

$$\mathbf{u}^e = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \end{bmatrix} = \begin{bmatrix} u_{iX} \\ u_{iY} \\ u_{iZ} \\ u_{jX} \\ u_{jY} \\ u_{jZ} \end{bmatrix}, \tag{3.5}$$

- its description in the local system

$$\mathbf{u'}^e = \begin{bmatrix} \mathbf{u'}_i \\ \mathbf{u'}_j \end{bmatrix} = \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{iz} \\ u_{jx} \\ u_{jy} \\ u_{jz} \end{bmatrix}. \tag{3.6}$$

- vector of the nodal forces of an element in the global system

$$\mathbf{f}^{e} = \begin{bmatrix} \mathbf{f}_i \\ \mathbf{f}_j \end{bmatrix} = \begin{bmatrix} F_{iX} \\ F_{iY} \\ F_{iZ} \\ F_{jX} \\ F_{jY} \\ F_{jZ} \end{bmatrix}, \tag{3.7}$$

and in the local system

$$\mathbf{f'}^{e} = \begin{bmatrix} \mathbf{f'}_i \\ \mathbf{f'}_j \end{bmatrix} = \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{iz} \\ F_{jx} \\ F_{jy} \\ F_{jz} \end{bmatrix}. \tag{3.8}$$

Interpretations and meanings of the used symbols can be found in Fig.3.2.

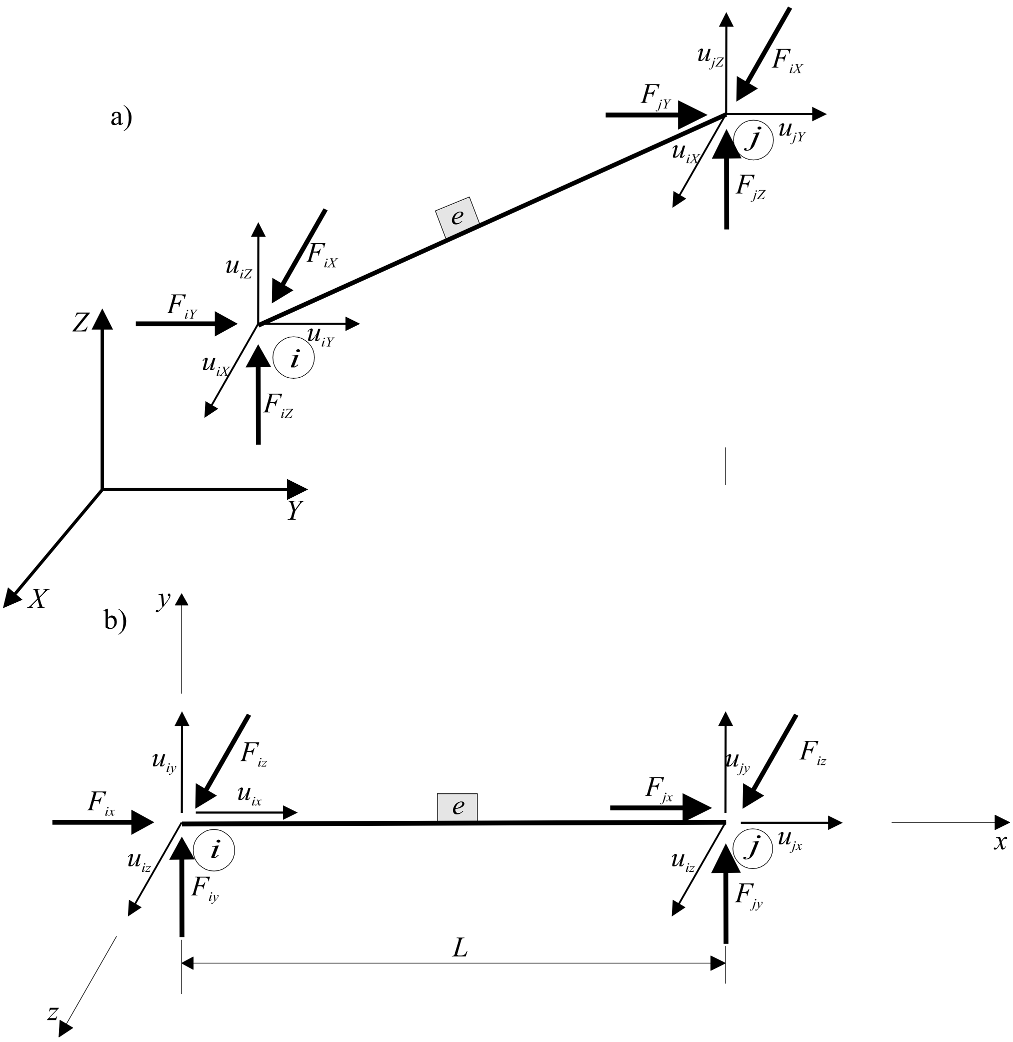

Fig.3.2

3.2. THE ELEMENT STIFFNESS MATRIX OF A SPACE TRUSS

The relation between nodal forces and nodal displacements for a space truss is identical to that for a plane truss if we analyse it in the local coordinate system. Obviously, the third force reaches F_{iz} or F_{jz} but the equilibrium equation of moments with respect to the y axis forces the zero value of this force:

$$
\begin{aligned}
&\text{a) } \sum F_x = F_{ix} + F_{jx} = 0 \rightarrow F_{ix} = -F_{jx}, \\
&\text{b) } \sum F_y = F_{iy} + F_{jy} = 0 \xrightarrow{\text{after considering eq. f}} F_{iy} = 0, \\
&\text{c) } \sum F_z = F_{iz} + F_{jz} = 0 \xrightarrow{\text{after considering eq. e}} F_{iz} = 0, \\
&\text{d) } \sum M_x = 0, \\
&\text{e) } \sum M_y = -F_{jz} L = 0 \rightarrow F_{jz} = 0, \\
&\text{f) } \sum M_z = -F_{jy} L = 0 \rightarrow F_{jy} = 0.
\end{aligned}
\tag{3.9}
$$

The relation between an axial force and displacements which is identical to the relation presented in Chapter II **Błąd! Nie można odnaleźć źródła odwołania.**) allows to express the searched dependence as follows:

$$\mathbf{f}'^{e} = \mathbf{K}'^{e}\,\mathbf{u}'^{e}, \tag{3.10}$$

where

$$\mathbf{K}'^{e} = \begin{bmatrix} \mathbf{J}' & -\mathbf{J}' \\ -\mathbf{J}' & \mathbf{J}' \end{bmatrix}, \tag{3.10a}$$

$$\mathbf{J}' = \frac{EA}{L}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.10b}$$

The transformation of the description of these equations from the local system to the global one will be done analogously to the transformation performed in case of a 2D truss (2.33), (2.34), (2.35).

In order to finish the transformation of the element stiffness matrix to the global system we need the rotation matrix of a node $\mathbf{R}_i$, and then we can determine components of the matrix $\mathbf{J}$ similar to those described by equation (2.37).

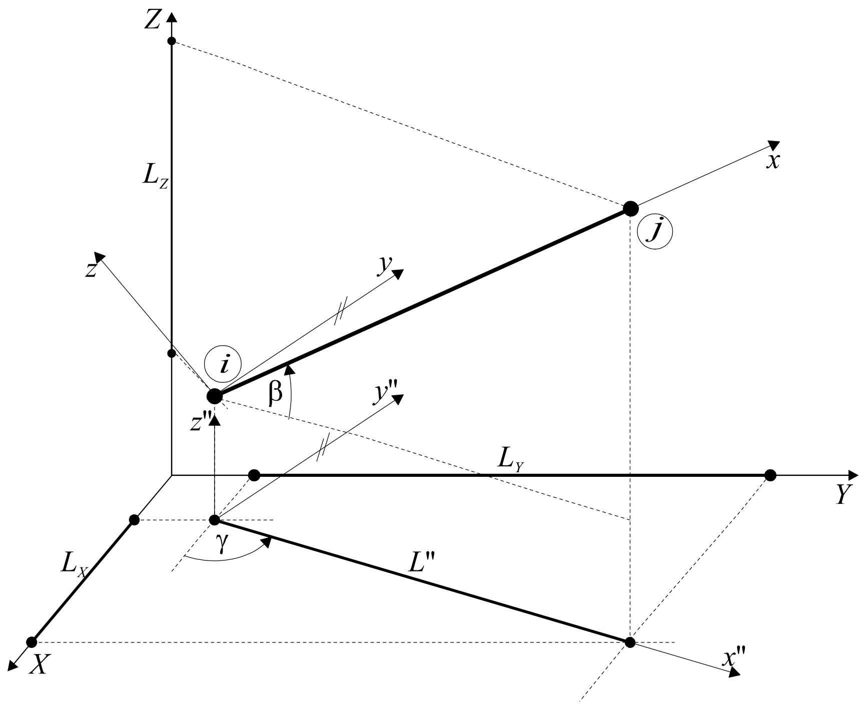

Fig.3.3

Since the location of the y and z axes of the local system is not essential for truss elements, we will choose the direction of the y axis in such a way that it will be always parallel to the plane XY of the global system but for bars parallel to the Z axis there will be an additional assumption that the y axis is parallel to the Y axis (comp.Fig.3.3).

The rotation from the local coordinate system to the global one will be composed of two intermediate rotations. First, we rotate the system xyz to the intermediate system $x''y''z''$ selected so that the x'' axis is parallel to the plane XY and next we rotate the system $x''y''z''$ by an angle γ so that the x'' and X axes are parallel. The first rotation around the y axis gives the following result:

$$\begin{bmatrix} u_{x''} \\ u_{y''} \\ u_{z''} \end{bmatrix} = \begin{bmatrix} c_\beta & 0 & -s_\beta \\ 0 & 1 & 0 \\ s_\beta & 0 & c_\beta \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix},$$

or in a shorter form $\mathbf{u}'' = \mathbf{R}_\beta\, \mathbf{u}'$, (3.11)

where $c_\beta = \cos\beta = \dfrac{L''}{L}$, $s_\beta = \sin\beta = \dfrac{L_Z}{L}$, $L_X = X_j - X_i$, $L_Y = Y_j - Y_i$, $L_Z = Z_j - Z_i$,

$L'' = \sqrt{L_X^2 + L_Y^2}$, $L = \sqrt{L''^2 + L_Z^2}$.

The second rotation around the z axis leads the relations to the global system:

$$\begin{bmatrix} u_X \\ u_Y \\ u_Z \end{bmatrix} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 \\ s_\gamma & c_\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x''} \\ u_{y''} \\ u_{z''} \end{bmatrix}$$

or in a shorter form $\mathbf{u} = \mathbf{R}_\gamma \, \mathbf{u}''$, (3.12)

where $c_\gamma = \cos\gamma = \dfrac{L_X}{L''}$, $s_\gamma = \sin\gamma = \dfrac{L_Y}{L''}$,

when L"=0 we assume γ=0, hence $c_\gamma = 1$ and $s_\gamma = 0$.

The composition of both rotations which means putting equation (3.11) into (3.12), gives the searched rotation matrix of a node

$$\mathbf{u}_i = \mathbf{R}_{i\gamma} \mathbf{R}_{i\beta} \mathbf{u}'_i, \tag{3.13}$$

where $\mathbf{R}_i = \mathbf{R}_{i\gamma} \mathbf{R}_{i\beta}$.

After multiplying matrices $\mathbf{R}_{i\gamma}$ $\mathbf{R}_{i\beta}$, we obtain the final form of the rotation matrix $\mathbf{R}_i$:

$$\mathbf{R}_i = \begin{bmatrix} c_\gamma c_\beta & -s_\gamma & -c_\gamma s_\beta \\ s_\gamma c_\beta & c_\gamma & -s_\gamma s_\beta \\ s_\beta & 0 & 0 \end{bmatrix}. \tag{3.14}$$

We calculate the transformation of the block **J** of the element stiffness matrix of the space truss from the local coordinate system to the global one analogically as in Chapter II (comp. the similar transformation of the stiffness matrix (2.34)).

$$\mathbf{J} = \mathbf{R}_i \mathbf{J}' \left(\mathbf{R}_i\right)^{\mathrm{T}}. \tag{3.15}$$

Inserting relations (3.10b) and (3.14) into the above equation we obtain:

$$\mathbf{J} = \frac{EA}{L} \begin{bmatrix} \left(c_\gamma c_\beta\right)^2 & c_\gamma s_\gamma \left(c_\beta\right)^2 & c_\gamma c_\beta s_\beta \\ c_\gamma s_\gamma \left(c_\beta\right)^2 & \left(s_\gamma c_\beta\right)^2 & s_\gamma c_\beta c_\beta \\ c_\gamma c_\gamma s_\beta & s_\gamma c_\beta s_\beta & \left(s_\beta\right)^2 \end{bmatrix}. \tag{3.16}$$

After the introduction of convenient notations:

$$C_X = \frac{L_X}{L}, \quad C_Y = \frac{L_Y}{L}, \quad C_Z = \frac{L_Z}{L} \tag{3.17}$$

which are called direction cosines of an element, we obtain a very simple form of the block **J** of the stiffness matrix:

$$\mathbf{J} = \frac{EA}{L}\begin{bmatrix} C_X^2 & C_X C_Y & C_X C_Z \\ C_X C_Y & C_Y^2 & C_Y C_Z \\ C_X C_Z & C_Y C_Z & C_Z^2 \end{bmatrix}. \tag{3.18}$$

Relation (3.18) obtained after being inserted into equation (3.10a) gives us the element stiffness matrix for the space truss in the global coordinate system.

3.3. THE VECTOR OF TEMPERATURE LOADS FOR AN ELEMENT OF 3D TRUSS

Since forming a loads vector of a truss for loads of concentrated forces is identical to forming it for a 2D truss, we will also neglect discussing the vector **p**. On the other hand, we will occupy with the vector of nodal forces due to a temperature load. Components of this vector in the local coordinate system are identical (apart from the correction in reference to the third component of the vector!) to the components of the vector for a plate truss (2.70).

$$\mathbf{f}'^{et} = EA\alpha_t \Delta t_o \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}. \tag{3.19}$$

The transformation to the global system proceeds in agreement with equation (2.31) in the following way:

$$\mathbf{f}^{et} = \mathbf{R}^e \, \mathbf{f}'^{et}, \tag{3.20}$$

where $\mathbf{R}^e$ is the element rotation matrix:

$$\mathbf{R}^e = \begin{bmatrix} \mathbf{R}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_j \end{bmatrix}. \tag{3.21}$$

Since a truss element is straight, then $\mathbf{R}_i$=$\mathbf{R}_j$, where the matrix $\mathbf{R}_i$ is defined by equation (3.14).

After making both the insertion of equation (3.14) into (3.20) and multiplication, we obtain

$$\mathbf{f}^{et} = EA\alpha_t \Delta t_o \begin{bmatrix} c_\gamma c_\beta \\ s_\gamma c_\beta \\ s_\beta \\ -c_\gamma c_\beta \\ -s_\gamma c_\beta \\ -s_\beta \end{bmatrix} \tag{3.22}$$

or another form:

$$\mathbf{f}^{et} = EA\alpha_t \Delta t_o \begin{bmatrix} C_X \\ C_Y \\ C_Z \\ -C_X \\ -C_Y \\ -C_Z \end{bmatrix}. \tag{3.23}$$

The further procedure is identical to the one employed in case of a plate truss.

3.4. THE BOUNDARY ELEMENT

In Chapter II we elaborated widely different types of boundary conditions and also elastic boundary elements. Since they are very useful elements by means of which we can model many different boundary conditions, we will pay more attention to them in this chapter concentrating on differences between plate and space trusses.

We will discuss the most general elastic element with stiffness k_b drooping with respect to axes of the global system with the angles α_X, α_Y, α_Z whose direction cosines are equal to

$$c_X = \cos\alpha_X,\ c_Y = \cos\alpha_Y,\ c_Z = \cos\alpha_Z. \tag{3.24}$$

The stiffness matrix of this element in the local system is analogous to the matrix stiffness of an ordinary truss element but this element has three degrees of freedom, so the stiffness matrix contains only one block $\mathbf{J'}$ (3.10)

$$\mathbf{K}'^b = k_b \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.25}$$

Transforming this element to the global coordinate system we obtain a matrix which is very like that one obtained in Chapter II for a plate truss:

$$\mathbf{K}^b = k_b \begin{bmatrix} c_X^2 & c_X c_Y & c_X c_Z \\ c_X c_Y & c_Y^2 & c_Y c_Z \\ c_X c_Z & c_Y c_Z & c_Z^2 \end{bmatrix} \tag{3.26}$$

Boundary elements can be composed of themselves forming for example, an element with three different types of stiffness k_x, k_y, k_z directed parallelly to axes of the local system xyz:

$$\mathbf{K}'^b = \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{bmatrix}. \tag{3.27}$$

The transformation of this matrix to the global system is analogous to the earlier written transformation of the block $\mathbf{J}'$ (3.15). We do not give the result of this transformation here leaving its execution as an exercise for the reader.

3.5. STRESSES AND INTERNAL FORCES

As in sec.2.11 of Chapter II we give here equations allowing to calculate stresses and internal forces in an element:

$$\sigma_x = E(\varepsilon - \varepsilon_t) = \frac{E}{L}\left[\left(u_{jx} - u_{ix}\right) - L\left(\alpha_t \Delta t_o\right)\right], \tag{3.28}$$

or in another form:

$$\sigma_x = \frac{E}{L}\begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{u}'^e - E\alpha_t \Delta t_o . \tag{3.29}$$

The transformation of the vector $\mathbf{u}'^e$ to the global system gives the relation:

$$\sigma_x = \frac{E}{L}\begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \left(\mathbf{R}^e\right)^{\mathrm{T}} \mathbf{u}^e - E\alpha_t \Delta t_o \tag{3.30}$$

which after making multiplication allows to write components of direct stress in an element as follows:

$$\sigma_x = E\left\{\begin{bmatrix} -\mathbf{c}^{\mathrm{T}} & \mathbf{c}^{\mathrm{T}} \end{bmatrix} \left(\mathbf{R}^e\right)^{\mathrm{T}} \mathbf{u}^e \frac{1}{L} - \alpha_t \Delta t_o\right\} \tag{3.31}$$

where $\mathbf{c}$ is the vector of element direction cosines: $\mathbf{c}^{\mathrm{T}} = \begin{bmatrix} c_X & c_Y & c_Z \end{bmatrix}$ (3.17).

Calculation of the normal force consists in to integrating stresses on the surface of a cross section with an assumption of homogeneity of the stress field (as we did in Chapter II)

$$N = \sigma_x A = EA\left\{\frac{1}{L}\begin{bmatrix}-\mathbf{c}^{\mathrm{T}} & \mathbf{c}^{\mathrm{T}}\end{bmatrix}\left(\mathbf{R}^e\right)^{\mathrm{T}}\mathbf{u}^e - \alpha_t \Delta t_o\right\}. \tag{3.32}$$

We calculate the remaining support reactions with the help of equation (2.75). We do it exactly in the same way as it has been done for the plate truss, so we will not describe here the above problem for a space truss in detail.