General Failure Criterion for Isotropic Media[a]

By Jerzy Podgórski[1]

Abstract: A general failure criterion dependent on three stress tensor invariants is proposed. It is applicable to a rather large class of materials including, metals, rocks, concrete and soils. The classical failure criteria and some recently proposed criteria are particular cases of this general criterion. The condition presented permits the uniform description of different groups of materials for which quite different forms of the failure criteria have been applied to date. A general form is applied to formulate failure criteria for plain concrete and sand. These criteria, to which smooth (conical for sand or paraboloidal for concrete) surfaces correspond, provide good agreement between predicted values of failure stresses and experimental results. This was possible due to the introduction of a new two-parameter function describing the deviatoric cross section of the failure surface. Two cross sectional shape characteristic ratios, λ and θ, defined in this paper, make possible the systematic analysis of different criteria and allow prediction of the failure surface features, which can be helpful in interpretation of the experimental results.

Introduction

Formulation of the exact failure criterion is still a problem for materials whose behavior is essentially dependent on the third stress tensor invariant and the hydrostatic pressure.

The classical Coulomb-Mohr criterion, often used for such media, gives results that differ considerably from test data, e.g., for rocks (2), sand (3,5,8,16), clay (10,18), concrete and mortar (1,6,12,17,19).

Because of these differences, many more attempts at precise determination of the failure criteria have been undertaken. For example, during the last decade, failure criteria had been formulated for sand by Gudehus (4), Lade and Duncan (9), and Matsuoka (11), and for concrete by Mills and Zimmerman (12), Willam and Warnke (20), Ottosen (14) and, recently, by Lade (7).

In this paper, a general failure criterion for isotropic media is proposed. This criterion is formulated in terms of three stress tensor invariants and includes both classical and recently proposed criteria as its particular cases.

A general form of the criterion is applied to formulate failure criteria for plain concrete and sand. These criteria, to which smooth conical (for sand) or paraboloidal (for concrete) surfaces correspond, provide good agreement between predicted values of failure stresses and experimental results.

Description of the Failure Surface

A simple description of the failure surface can be made by using the Heigh-Westergaard cylindrical coordinate system (Fig. 1), related to the cartesian coordinate system of principal stresses by the following equations:

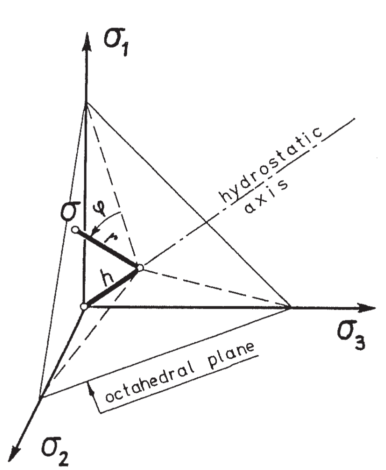

FIG. 1.—Two Coordinate Systems in Space of Principal Stresses

$$\left.\begin{aligned} r &= \sqrt{2J_2} = \tau_0\sqrt{3} \\ \cos 3\varphi &= J = \frac{3\sqrt{3}J_3}{2(J_2)^{3/2}} \\ h &= \frac{I_1}{\sqrt{3}} = \sigma_0\sqrt{3} \end{aligned}\right\} \qquad (1)$$

in which $I_1 = \sigma_{ii}$ is a first invariant of the stress tensor; $\sigma_0 = 1/3\ I_1$ is a mean stress (tension is positive); $J_2 = 1/2\ s_{ij}\ s_{ij}$ is the second invariant of deviatoric part of the stress tensor ($s_{ij} = \text{dev}\ \sigma_{ij}$); $\tau_0 = \sqrt{2/3\,J_2}$ = octahedral shear stress; and $J_3 = 1/3\ s_{ij}\ s_{jk}\ s_{ki}$ = third invariant of stress deviator tensor.

Trace of a Failure Surface in Deviatoric Plane

Experimental results show that the shape of the deviatoric cross section of the failure surface is very important for good agreement of predicted and experimental results. This fact led to the proposal of many different functions on which the shape of the cross section might depend. In this paper, these functions, in the form $r = r(\varphi)$, will be called shape functions.

Two simple forms of these functions have been proposed, one by Mills and Zimmerman (12):

$$r = r_0 - J \qquad (2)$$

and one by Gudehus (4):

$$r^2 = r_0 - J \qquad (3)$$

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in which $J = \cos 3\varphi$ and r_0 = const. must satisfy the convexity condition, $r_0 \geq 10$, for the Mills, Zimmerman function, and $r_0 \geq 4$ for the Gudehus function.

Willam and Warnke (20) use the ellipse equation:

$$r = \frac{2(1-\lambda^2)\cos\varphi + (2\lambda - 1)\sqrt{4(1-\lambda^2)\cos^2\varphi + 5\lambda^2 - 4\lambda}}{4(1-\lambda^2)\cos\varphi^2 + (2\lambda-1)^2} \quad \text{(4)}$$

in which λ is some constant, i.e.:

$$\lambda = \frac{r(\varphi = 0)}{r\left(\varphi = \dfrac{\pi}{3}\right)} \quad \text{(5)}$$

and $0, 5 \leq \lambda \leq 2$ for every convex curve.

The Ottosen (14), Lade and Duncan (7,9) and Matsuoka (11) failure criteria take the form of a shape function:

$$r = \left[\cos\left(\frac{1}{3}\arccos \alpha J\right)\right]^{-1} \quad \text{(6)}$$

in which α = const., satisfying the condition $0 \leq \alpha \leq 1$.

The classical Coulomb-Mohr criterion gives

$$r = \left[\cos\left(\frac{1}{3}\arccos J - \beta\right)\right]^{-1} \quad \text{(7)}$$

in which β = const. and depends on the angle of internal friction (see Table 1).

Shape functions presented in Eqs. 2–4, 6 and 7 lead to failure criteria for which a perfect agreement with test results is not possible, because only one shape characteristic ratio, e.g., λ can be determined independently and all other characteristics depend on it. For example, when the shape characteristic λ is assumed to be given, then the ratio θ defined by

$$\theta = \frac{r\left(\varphi = \dfrac{\pi}{6}\right)}{r\left(\varphi = \dfrac{\pi}{3}\right)} \quad \text{(8)}$$

is dependent on the chosen value of λ.

Fig. 2 shows θ-λ characteristics for shape functions described by Eqs. 2–4, 6 and 7, and some points obtained experimentally for concrete, sand and clay. Comparison between these experimental points and the curves given by the shape functions suggests that a two-parameter shape function would describe the experimental results much better.

Consider the following two-parameter function:

$$r = \frac{1}{P(J)}, \quad P(J) = \cos\left(\frac{1}{3}\arccos \alpha J - \beta\right) \quad \text{(9)}$$

TABLE 1.—Some Particular Cases of Criterion Given by Eq. 13

Shape of failure surface (1)	Failure criteria (2)	Functions A_0 (3)	A_1 (4)	A_2 (5)	Remarks (6)
Cylindrical	Huber-Mises $J_2 = k^2$	$-C_0$	1	0	Circular cylinder, $C_0 = \sqrt{2/3}\,k$
	Tresca $\lvert\tau_{max}\rvert = k$	$-C_0$	$P(J)$	0	Hexagonal prism, $C_0 = \sqrt{2/3}\,k$, $\alpha = 1$, $\beta = \pi/6$
Conical	Drucker-Prager $\sqrt{J_2} = a - b\,I_1$	$C_1\sigma_0 - C_0$	1	0	Circular cone, $C_0 = \sqrt{2/3}\,a$, $C_1 = \sqrt{6}\,b$
	Lade-Duncan (9) $I_1^3/I_3 = \kappa_1$	$C_1\sigma_0$	$P(J)$	0	$C_1 = \sqrt{(\kappa_1 - 27)/(2\kappa_1)}$, $\alpha = \sqrt{(\kappa_1 - 27)/\kappa_1}$, $\beta = 0$
	Matsuoka (11) $I_1I_2/I_3 = 8(K^2 + 1)$	$C_1\sigma_0$	$P(J)$	0	$C_1 = \sqrt{3/(6 + 4/K^2)}$ $\alpha = (1 + 1/K^2)/\sqrt{1 + 2/(3K^2)}$ $\beta = 0$
	Coulomb-Mohr $\lvert\tau_n\rvert = c - \sigma_n \tan\phi$	$C_1\sigma_0 - C_0$	$P(J)$	0	Hexagonal pyramid, $C_0 = (\sqrt{2}\,c\cos\phi)/\sqrt{3 + \sin^2\phi}$, $C_1 = (\sqrt{2}\sin\phi)/\sqrt{3 + \sin^2\phi}$, $\alpha = 1$, $\tan\beta = \sqrt{3}\,(1 - \sin\phi)/(1 + \sin\phi)$
Paraboloidal	Similar to Willam-Warnke (20) failure criterion for concrete	$C_0 - \sigma_0$	$C_1P(J)$	$C_2[P(J)]^2$	Paraboloid with constant shape of deviatoric cross section
	Ottosen (14) $AJ_2/f_c^2 + P\sqrt{J_2}/f_c + BI_1/f_c - 1 = 0$ in which $P = K_1 \cos(1/3 \arccos K_2J)$	$C_0 - \sigma_0$	$C_1P(J)$	C_2	Paraboloid with changeable shape of deviatoric cross section, $\beta = 0$

Note: $I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3$ = second stress tensor invariant; $I_3 = \sigma_1\sigma_2\sigma_3$ = third stress tensor invariant; k, a, b, κ_1, K, c, A, B, K_1 and K_2 = constants; ϕ = angle of internal friction; C_0, C_1 and C_2 = constants; and f_c = uniaxial compressive strength.

in which α = const. and β = const., which satisfy the conditions $0 \leq \alpha \leq 1$, $0 \leq \beta \leq \pi/6$. Eq. 9 describes a smooth convex curve, part of which is shown in Fig. 3(*a*).

In some domain of the θ-λ plane (see Fig. 4), the complicated form of Eq. 9 may be replaced by the simpler:

$$r = \frac{1}{P'(J)}, \quad P'(J) = 1 + c\sqrt{d + J} \quad \text{(10)}$$

in which $d \geq 1$ and c = constants which have to satisfy the convexity condition $c \leq (2\sqrt{d+1})/(7 - 2d)$. Fig. 3(*b*) shows the shape of the curve described by Eq. 10, which allows enlargement of the domain in which the failure criterion may be used (see the dashed line in Fig. 4).

The θ-λ characteristics of the functions given by Eqs. 9 and 10 are shown in Fig. 4, in which the solid line limits the region described by the α and β parameters of Eq. 9, and the dashed line limits the region described

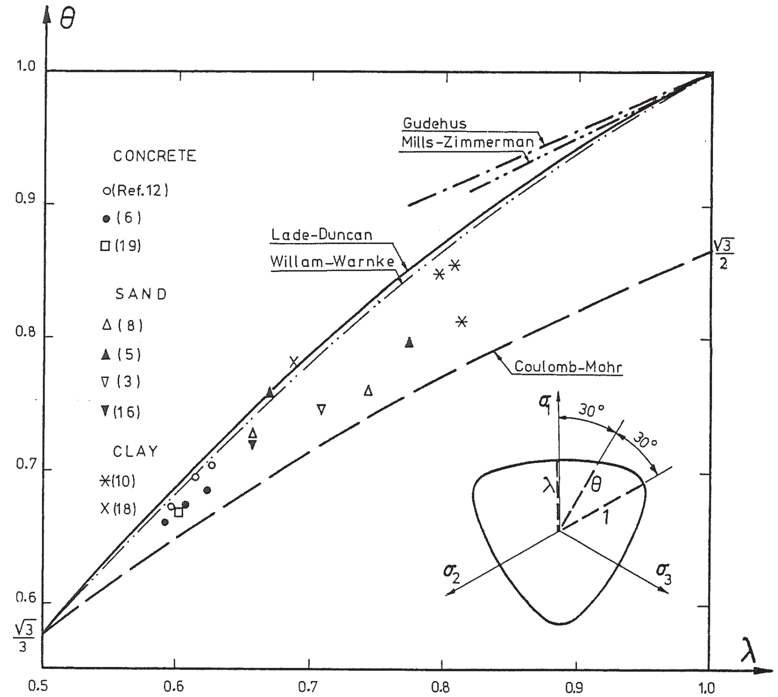


FIG. 2.—θ-λ Characteristics for Different Shape Functions

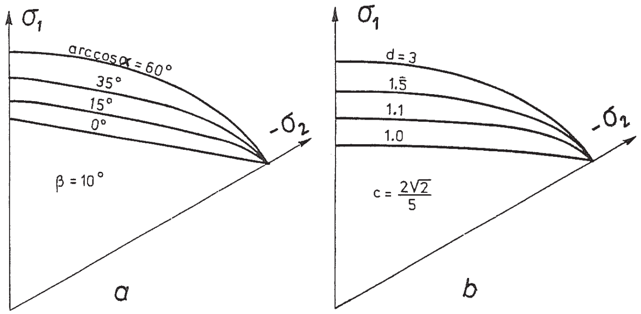


FIG. 3.—Shape of Curves Given by: (*a*) Eq. 9; (*b*) Eq. 10

by the c and d parameters of Eq. 10.

A comparison of Figs. 2 and 4 shows that the functions introduced provide good agreement with tests results for concrete, sand and clay because all experimental points from Fig. 2 are included in the region described by the parameters of Eqs. 9 and 10 (Fig. 4).

The value of parameters α and β in Eq. 9 may be determined by the following equations:

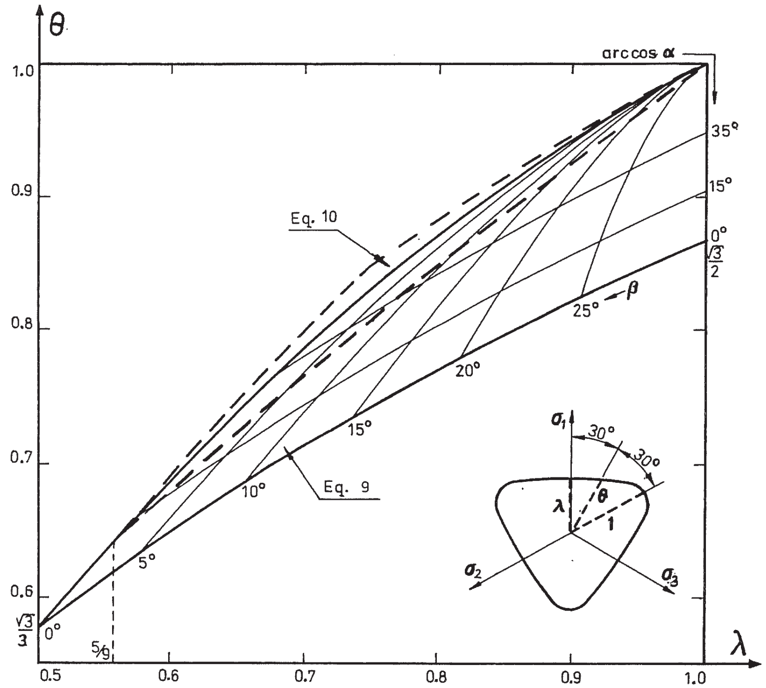


FIG. 4.—θ-λ Characteristics for Proposed Shape Functions

$$\left.\begin{aligned} \alpha &= \cos 3x \\ \tan x &= \frac{\lambda \cos \beta - \cos\left(\dfrac{\pi}{3} - \beta\right)}{\sin\left(\dfrac{\pi}{3} - \beta\right) - \lambda \sin \beta} \\ \tan \beta &= \frac{2\lambda \cos x - \sqrt{3}\,\theta}{\theta - 2\lambda \sin x} \end{aligned}\right\} \qquad (11)$$

using an iterative method in which the initial value of β can be taken as equal to zero, or from the diagram plotted in Fig. 4.

Parameters c and d, occurring in Eq. 10, can be evaluated from

$$\left.\begin{aligned} d &= 1 + \frac{e^2 - 2}{3 - 2e} \\ e &= \frac{(1-\lambda)\theta}{2(1-\theta)\lambda} + \frac{(1-\theta)}{(1-\lambda)} \\ c &= \frac{1-\lambda}{\lambda\sqrt{d+1} - \sqrt{d-1}} \end{aligned}\right\} \qquad (12)$$

in which values of characteristic ratios λ and θ must be calculated on the basis of experimental data (see examples of application).

Formulation of General Failure Criterion

The general failure criterion for isotropic materials may be expressed in the following form:

$$A_0 + A_1\tau_0 + A_2\tau_0^2 = 0 \quad (13)$$

in which A_0 is a function of hydrostatic pressure only; and A_1 and A_2 = functions dependent on $J = \cos 3\varphi$ or on the shape functions (Eq. 9 or 10).

This general form of the failure criterion permits the creation of many different criteria. Table 1 shows some particular cases of the failure criteria and specification of the A_0, A_1 and A_2 functions in these cases.

Examples of Application of General Criterion

Failure Criterion for Concrete.—In this case the best form of the failure criterion is

$$\sigma_0 - C_0 + C_1P\tau_0 + C_2\tau_0^2 = 0 \quad (14)$$

in which $P = \cos(1/3 \text{ arc cos } \alpha J - \beta)$ and C_0, C_1 and C_2 = const. The failure surface corresponding to Eq. 14 satisfies the following conditions formulated by Newman and Newman (13) and by Ottosen (14): (1) Failure surface is convex, curved and smooth; (2) radius r of the deviatoric cross section of the failure surface increases with increasing hydrostatic pressure; and (3) characteristic ratio λ changing from the value $\lambda \geq 0.5$ near the vertex of the failure surface ($\tau_0 \to 0$) to $\lambda = 1$ for very high hydrostatic pressure ($\sigma_0 \to -\infty$).

The criterion in the form given by Eq. 14 contains five parameters, C_0, C_1, C_2, α and β, the values of which can be determined on the basis of some simple test data (see Table 2).

The following relations may be adopted on the basis of the Paul (15) tension cut-off hypothesis and the results of tests performed by Andenaes et al. (1), Kupfer (6) and Tasuji et al. (19) (for notation see Table 2):

$$f_{vt} = f_t, \quad f_{cc} = 1.1 f_c, \quad f_{oc} = 1.25 f_c$$

Characteristic ratios λ and β and values of parameters α and β have been calculated for these relations and presented in Table 3.

TABLE 2.—Tests Used for Parameter Determination in Failure Criterion for Concrete

Type of test (1)	$\sigma_1 : \sigma_2 : \sigma_3$ (2)	Strength-Max σ_1 (3)	Remarks (4)
Uniaxial tension	1 : 0 : 0	f_t	$J = 1$
Uniaxial compression	1 : 0 : 0	f_c	$J = -1$
Biaxial compression	1 : 0.5 : 0	f_{oc}	$J = 0$
Biaxial compression	1 : 1 : 0	f_{cc}	$J = 1$
Triaxial tension	1 : 1 : 1	f_{vt}	Vertex

TABLE 3.—Characteristic Ratios λ and θ, and Values of arc cos α and β Parameters versus f_t/f_c

f_t/f_c (1)	λ (2)	θ (3)	arc cos α, in degrees (4)	β, in degrees (5)
0.06	0.51375	0.59182	2.034	0.235
0.07	0.51567	0.59386	2.339	0.261
0.08	0.51748	0.59581	2.635	0.283
0.09	0.51917	0.59764	2.922	0.300
0.10	0.52074	0.59936	3.197	0.313
0.11	0.52219	0.60097	3.462	0.321
0.12	0.52351	0.60246	3.717	0.325

Values of parameters C_0, C_1 and C_2 can be calculated using the equations:

$$C_0 = f_t \quad (15)$$

$$C_1 = \frac{\sqrt{2}}{P_0}\left(1 - \frac{3}{2}\,\frac{\dfrac{f_t}{f_{cc}}}{\dfrac{f_{cc}}{f_t} - 1}\right) \quad (16)$$

$$C_2 = \frac{9}{2}\,\frac{\dfrac{f_t}{f_{cc}}}{f_{cc} - f_t} \quad (17)$$

in which $P_0 = P(\varphi = 0) = \cos(1/3 \text{ arc cos } \alpha - \beta)$.

Figs. 5 and 6 show the failure envelope in a biaxial state of stress and the failure surface cross section in the τ_0-σ_0 plane ($J = \pm 1$). The proposed criterion is compared with that of Ottosen (14) and Lade (7) on the basis of test results obtained by Andenaes et al. (1), Kupfer (6), Mills and Zimmerman (12), Schickert and Winkler (17) and Tasuji et al. (19). This comparison shows that the concrete behavior is described more realistically in the region $J = 0$ ($\sigma_1/\sigma_2 = 0.5$ in biaxial state of stress) by the proposed criterion than by the previous criteria.

Failure Criterion for Sand.—Failure criteria for cohesionless media may usually be described with sufficient precision by a linear version of Eq. 13:

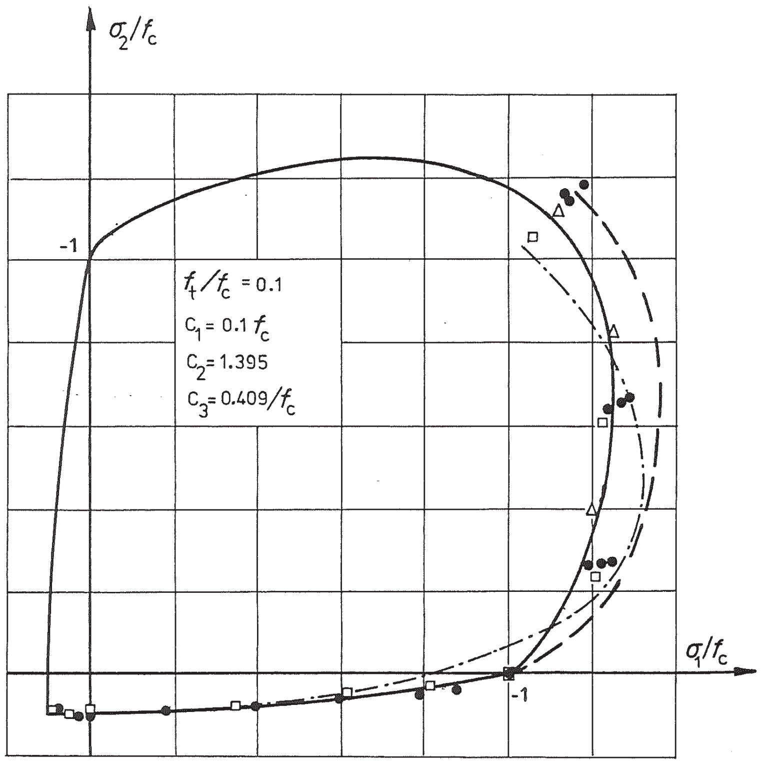
$$\sigma_0 - C_0 + C_1P\,\tau_0 = 0 \quad (18a)$$

$$\text{or} \quad \sigma_0 - C_0 + C_1P'\,\tau_0 = 0 \quad (18b)$$

In failure surface equations for dense sand and clay, the simple form of the shape function given by Eq. 10 may be used, as it makes all calculations easier.

The method of determining the four parameters C_0, C_1, α and β occuring in Eq. 18 may be exemplified by using the test results of Green and Bishop for sand (3).

Table 4 shows values of the internal friction angle determinated on



Solid Line - Proposed Criterion; Dashed Line - Ottosen Criterion (14) ; Dot Line - Lade Criterion (7); Tests Results of Kupfer (6) ● ; Tasuji, et al. (19) □ ; Andenaes, et al. (1) Δ

FIG. 5.—Failure Envelope for Concrete in Biaxial Stresses

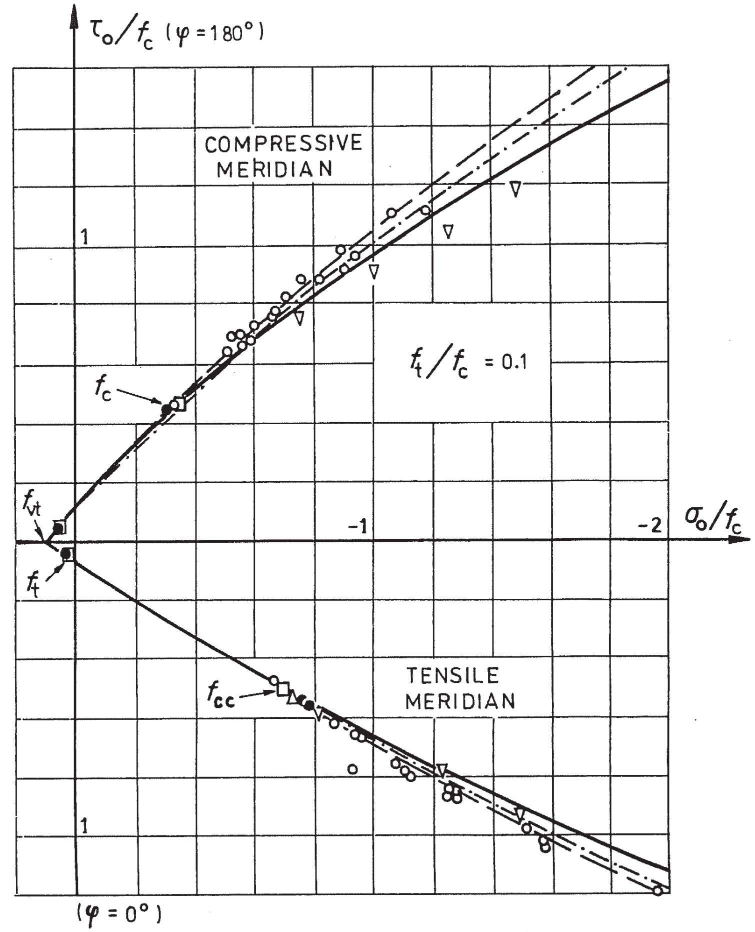
the basis of Fig. 3 from the Green and Bishop paper (3). Using this data and assuming zero cohesion, the ratios λ and θ may be determined from

$$\lambda = \frac{\dfrac{3}{\sin \phi_c} - 1}{\dfrac{3}{\sin \phi_t} + 1}, \quad \theta = \frac{\sin \phi_0}{2\sqrt{3}}\left(\frac{3}{\sin \phi_c} - 1\right) \quad \text{(19)}$$

in which ϕ_c, ϕ_0 and ϕ_t = angles of internal friction for triaxial compression ($J = -1$), triaxial shear ($J = 0$) and triaxial tension ($J = 1$), respectively.

For cohesionless media, $C_0 = 0$ and the α and β parameters may be obtained (having determined λ and θ) from Eq. 11.

Thus, in Eq. 18, only one parameter C_1 must be known. The appropriate equation is



Solid Line - Proposed Criterion; Dashed Line - Ottosen Criterion (14); Dot Line - Lade Criterion (7); Tests Results of Mills and Zimmerman (12) ○ ; Kupfer (6) ● ; Tasuji, et al. (19) □ ; Andenaes, et al. (1) Δ ; Schickert and Winkler (17) ∇

FIG. 6.—Failure Surface Cross Section in τ_0-σ_0 Plane

$$C_1 = 2\sqrt{2}\,\frac{P_0}{\dfrac{3}{\sin \phi_0} + 1}$$

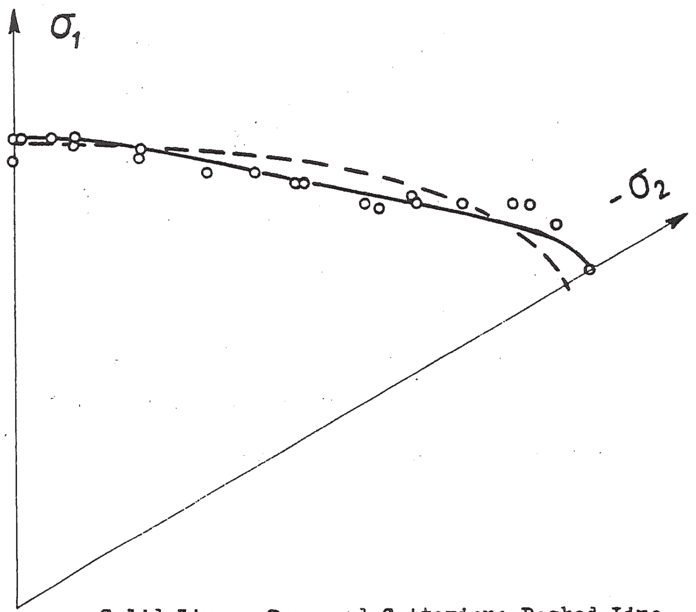
in which $P_0 = \cos(1/3 \text{ arc} \cos \alpha - \beta)$.

In Table 4, $\phi_t = 44°$; $\phi_c = 39°$; and ϕ_0 may be taken as $= 43.5°$ so that $\lambda = 0.7083$; $\theta = 0.7486$; $\alpha = 0.9721$; $\beta = 10.28°$; and $C_1 = 0.5291$.

TABLE 4.—Internal Friction Angles, ϕ, for Sand

$b = \sigma_2 - \sigma_3/\sigma_1 - \sigma_3$[a] (1)	φ = arc tan $[\sqrt{3}(1-b)/(1+b)]$, in degrees (2)	ϕ, in degrees (3)
0	60	39
0.09	55.33	42.5
0.14	52.57	44
0.16	51.43	44
0.22	47.92	43.5
0.27	44.87	43.5
0.28	44.25	44
0.31	42.37	42
0.33	41.11	42
0.43	34.62	43
0.44	33.96	43
0.51	29.34	43.5
0.59	24.07	43
0.72	15.75	45, 43.5
0.86	7.43	45, 44
0.91	4.67	45
0.98	1	44
1	0	44, 41

[a] $\sigma_1 > \sigma_2 > \sigma_3$.



Solid Line - Proposed Criterion; Dashed Line - Lade and Duncan Criterion (9); Test Data of Green and Bishop (3) o

FIG. 7.—Deviatoric Cross Section of Failure Surface for Sand

In Fig. 7, the solid line shows the deviatoric cross section of the failure surface determined by these parameters. The points occurring in Fig. 7 represent the Green and Bishop test data, whereas the dashed line shows the Lade and Duncan failure criterion (9) at $\kappa_1 = 56$ (see Table 1).

Good agreement between the proposed criterion and the experimental results can be observed, especially in the region where $J = 0$ ($\varphi = 30°$), possible because of the two-parameter shape function, whereas in other criteria (Lade and Duncan, Coulomb-Mohr), one-parameter shape functions used do not attain such close agreement.

Summary and Conclusions

A general failure criterion dependent on three stress tensor invariants is proposed. It is applicable to a rather large class of materials including, e.g., metals, rocks, concrete and soils.

The classical Huber-Mises, Tresca, Coulomb-Mohr, Drucker-Prager criteria, and those proposed more recently by Lade and Duncan (9) and Ottosen (14), are some particular cases of the general criterion presented here.

The condition presented permits the uniform description of different groups of materials for which quite different forms of failure criteria have been applied to date.

The five-parameter form of the criterion used to determine the fracture condition for concrete, and the three-parameter form used as a failure criterion for sand, provide perfect agreement between the theoretical prediction of the limiting stress and the experimental results. This was possible due to introduction of a new two-parameter function describing the shape of the deviatoric cross section of the failure surface.

Two cross sectional shape characteristic ratios defined in this paper, λ and θ, make possible the systematic analysis of different criteria and allow prediction of the failure surface features, which can be helpful in interpretation of the experimental results.

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Appendix II.—Notation

The following symbols are used in this paper:

A_0, A_1, A_2 = some functions in a general form of the failure criterion, Eq. 13;
C_0, C_1, C_2 = parameters in Eq. 14 or Eq. 15;

$f_t, f_c, f_{0c}, f_{cc}, f_{vt}$ = failure stresses for concrete (see Table 2);
I_1 = σ_{kk}, first invariant of the stress tensor;
I_2 = $\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3$, second invariant of the stress tensor;
I_3 = $\sigma_1\sigma_2\sigma_3$, third invariant of the stress tensor;
J = $\cos 3\varphi$, invariant of the stress deviator;
J_2 = $(1/2)s_{ij}s_{ij}$, second invariant of the stress deviator;
J_3 = $(1/3)s_{ij}s_{jk}s_{ki}$, third invariant of the stress deviator;
$k, a, b, c, \kappa_1, K, A, B, K_1, K_2$ = parameters in some failure criteria (see Table 1);
P, P' = functions on which the shape of the deviatoric cross section of the failure surface is dependent, Eqs. 9 and 10;
r, φ, h = cylindrical coordinates in the space of the principal stresses, defined by Eq. 1;
s_{ij} = dev σ_{ij}, deviatoric part of the stress tensor;
α, β, c, d = parameters of the shape functions, Eqs. 9 and 10;
λ, θ = characteristics of the deviatoric cross section of the failure surface, defined by Eqs. 5 and 8;
σ_{ij} = stress tensor;
σ_0 = $(1/3)I_1$, mean stress (tension is positive);
$\sigma_1, \sigma_2, \sigma_3$ = principal stresses;
τ_0 = $\sqrt{(2/3)J_2}$, octahedral shear stress;
ϕ = angle of internal friction; and
ϕ_c, ϕ_0, ϕ_t = angles of internal friction at triaxial compression, shear and tension, respectively.