ADVANCED TOPICS IN FINITE ELEMENT METHOD

▪ Dynamics

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DISCRETE STRUCTURAL MECHANICS EXPRESSED AS FORCE BALANCE STATEMENTS

Case	*Problem type*	*Governing force balance equations*
I	General nonlinear dynamics	$\underbrace{\mathbf{p}(\mathbf{u}, \dot{\mathbf{u}}, \ddot{\mathbf{u}}, t)}_{internal} = \underbrace{\mathbf{f}(\mathbf{u}, \dot{\mathbf{u}}, t)}_{external}$
II	General nonlinear statics	$\underbrace{\mathbf{p}(\mathbf{u})}_{internal} = \underbrace{\mathbf{f}(\mathbf{u})}_{external}$
III	Flexible structure nonlinear dynamics	$\underbrace{\mathbf{p}_i(\mathbf{u}, \dot{\mathbf{u}}, \ddot{\mathbf{u}}, t)}_{inertial} + \underbrace{\mathbf{p}_d(\mathbf{u}, \dot{\mathbf{u}}, t)}_{damping} + \underbrace{\mathbf{p}_e(\mathbf{u}, t)}_{elastic} = \underbrace{\mathbf{f}(\mathbf{u}, t)}_{external}$
IV	Flexible structure linear dynamics	$\underbrace{\mathbf{M}\,\ddot{\mathbf{u}}(t)}_{inertial} + \underbrace{\mathbf{C}\,\dot{\mathbf{u}}(t)}_{damping} + \underbrace{\mathbf{K}\,\mathbf{u}(t)}_{elastic} = \underbrace{\mathbf{f}(t)}_{external}$
V	Linear elastostatics	$\underbrace{\mathbf{K}\,\mathbf{u}}_{elastic} = \underbrace{\mathbf{f}}_{external}$
VI	Dynamic perturbations	$\underbrace{\mathbf{M}(\mathbf{u})\,\ddot{\mathbf{d}}(t)}_{inertial} + \underbrace{\mathbf{C}(\mathbf{u})\,\dot{\mathbf{d}}(t)}_{damping} + \underbrace{\mathbf{K}(\mathbf{u})\,\mathbf{d}(t)}_{elastic} + \underbrace{\mathbf{p}(\mathbf{u}) = \mathbf{f}(\mathbf{u})}_{static\ equilibrium}$

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VI	Dynamic perturbations	$\underbrace{\mathbf{M}(\mathbf{u})\,\ddot{\mathbf{d}}(t)}_{inertial} + \underbrace{\mathbf{C}(\mathbf{u})\,\dot{\mathbf{d}}(t)}_{damping} + \underbrace{\mathbf{K}(\mathbf{u})\,\mathbf{d}(t)}_{elastic} + \underbrace{\mathbf{p}(\mathbf{u}) = \mathbf{f}(\mathbf{u})}_{static\ equilibrium}$
VII	Damped forced vibrations	$\underbrace{\mathbf{M}\,\ddot{\mathbf{u}}(t)}_{inertial} + \underbrace{\mathbf{C}\,\dot{\mathbf{u}}(t)}_{damping} + \underbrace{\mathbf{K}\,\mathbf{u}(t)}_{elastic} = \underbrace{\mathbf{f}_p(t)}_{periodic}$
VIII	Damped free vibrations	$\underbrace{\mathbf{M}\,\ddot{\mathbf{u}}(t)}_{inertial} + \underbrace{\mathbf{C}\,\dot{\mathbf{u}}(t)}_{damping} + \underbrace{\mathbf{K}\,\mathbf{u}(t)}_{elastic} = \mathbf{0}$
IX	Undamped free vibrations	$\underbrace{\mathbf{M}\,\ddot{\mathbf{u}}(t)}_{inertial} + \underbrace{\mathbf{K}\,\mathbf{u}(t)}_{elastic} = \mathbf{0}$

Symbol $\mathbf{u}$ is array of total displacement DOFs; $\mathbf{d}$ in case VI is a linearized perturbation of $\mathbf{u}$. Symbol t denotes time. Superposed dots abbreviate time derivatives: $\dot{\mathbf{u}} = d\mathbf{u}/dt$, $\ddot{\mathbf{u}} = d^2\mathbf{u}/dt^2$, etc. The history $\mathbf{u} = \mathbf{u}(t)$ is called the *response* of the system. This term is extendible to nonlinear statics. Initial force effects $\mathbf{f}_I$ may be accommodated in forced cases by taking $\mathbf{f} = \mathbf{f}_I$ when $\mathbf{u} = \mathbf{0}$.

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UNDAMPED FREE VIBRATIONS

- Differential equation describing free vibrations of a straight bar using the Bernoulli model (without transverse forces and rotational inertia):

$$EJ \cdot \frac{\mathrm{d}^4}{\mathrm{d}x^4} u(x,t) + \mu \cdot \frac{\mathrm{d}^2}{\mathrm{d}t^2} u(x,t) = 0$$

- By adopting a solution in the form:

$$u(x,t) = y(x) \cdot sin(\omega \cdot t)$$

we will get:

$$\left[\frac{\mathrm{d}^4}{\mathrm{d}x^4} y(x) - \frac{\mu \cdot \omega^2}{EJ} \cdot y(x) \right] \cdot sin(\omega \cdot t) = 0$$

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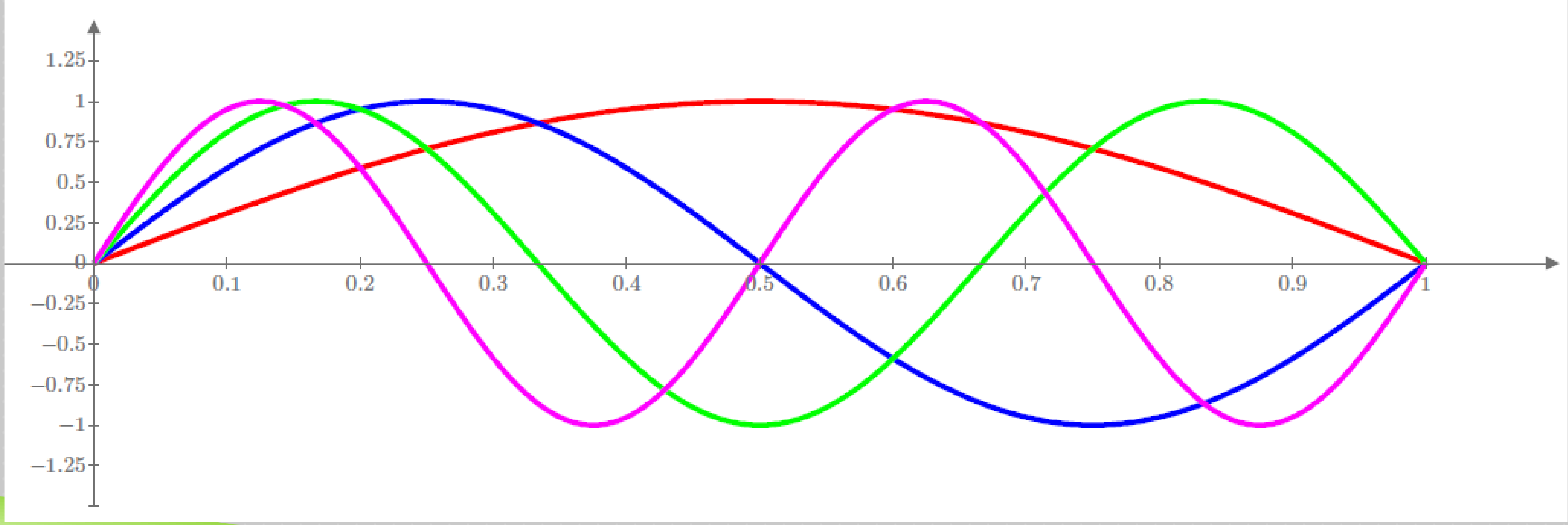
UNDAMPED FREE VIBRATIONS

- After taking into account the boundary conditions, this gives the following forms of natural vibrations:

$$y(0)=0 \qquad y(L)=0$$

$$M(0)=0 \qquad M(L)=0$$

$$\frac{M(x)}{EJ}=\frac{\mathrm{d}^2}{\mathrm{d}x^2}y(x)$$



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UNDAMPED FREE VIBRATIONS

- Writing the equation equation in matrix we have:

$$\mathbf{M}\,\ddot{\mathbf{u}}(t) + \mathbf{K}\,\mathbf{u}(t) = \mathbf{0}$$

where $\mathbf{M}$ is the inertia matrix, $\mathbf{K}$ is the stiffness matrix and $\mathbf{u}$ is the displacement vector.

- Substituting $\mathbf{u}(t) = \mathbf{v}\,e^{j\omega t}$ we will get:

$$(-\omega^2\mathbf{M} + \mathbf{K})\,\mathbf{v} = \mathbf{0} \quad \text{or} \quad \mathbf{K}\,\mathbf{v} = \omega^2\mathbf{M}\mathbf{v}$$

- For beam element

$$\mathbf{K}^{\mathrm{e}} = \frac{EJ}{Le^2}\cdot\begin{bmatrix} \frac{12}{Le} & 6 & \frac{-12}{Le} & 6 \\ 6 & 4\cdot Le & -6 & 2\cdot Le \\ \frac{-12}{Le} & -6 & \frac{12}{Le} & -6 \\ 6 & 2\cdot Le & -6 & 4\cdot Le \end{bmatrix}$$

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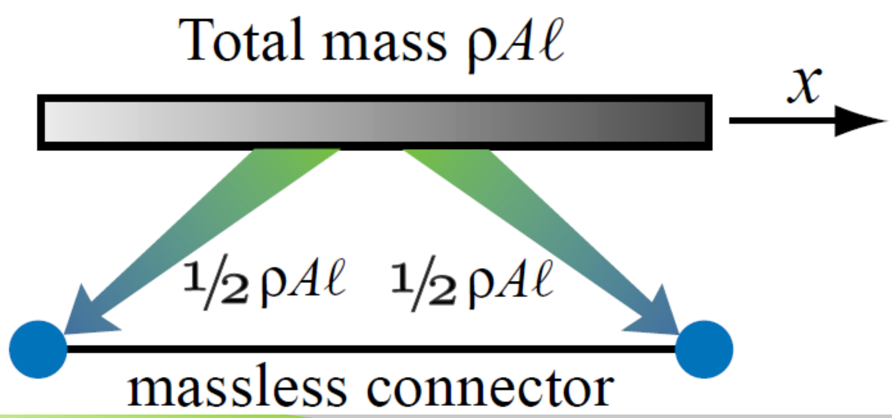
MASS MATRIX CONSTRUCTION

- **Direct Mass Lumping**
- The total mass of element e is directly apportioned to nodal freedoms, ignoring any cross coupling. The goal is to build a diagonally lumped mass matrix (DLMM) $\mathbf{M}_{\mathrm{L}}$
- As the simplest example, consider a 2-node prismatic bar element with length l, cross section area A, and mass density ρ, which can only move in the axial direction x:



MASS MATRIX CONSTRUCTION

- The total mass of the element is $m=\rho Al$.
- This is divided into two equal parts and assigned to each end node to produce: $\mathbf{M}_L^e = \frac{1}{2}\rho A\ell \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}\rho A\ell\, \mathbf{I}_2$
- Kinetic energy of the element is:

$$T^e = \frac{1}{2}(\dot{\mathbf{u}}^e)^T \mathbf{M}_L^e \dot{\mathbf{u}}^e = \frac{1}{2}\rho A\ell\, v^2 = \frac{1}{2} M^e v^2$$

$$\vec{\mathbf{v}}^e = \mathbf{N}_v^e \dot{\mathbf{u}}^e$$

$$T^e = \frac{1}{2}(\dot{\mathbf{u}}^e)^T \int_{\Omega^e} \rho (\mathbf{N}_v^e)^T \mathbf{N}_v^e \, d\Omega\, \dot{\mathbf{u}}^e \stackrel{\text{def}}{=} \frac{1}{2}(\dot{\mathbf{u}}^e)^T \mathbf{M}^e \dot{\mathbf{u}}^e$$

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MASS MATRIX CONSTRUCTION

- Element mass matrix follows as the Hessian of T^e

$$\mathbf{M}^e = \frac{\partial^2 T^e}{\partial \dot{\mathbf{u}}^e \partial \dot{\mathbf{u}}^e} = \int_{\Omega^e} \rho \left(\mathbf{N}_v^e\right)^T \mathbf{N}_v^e \, d\Omega$$

- If the same shape functions used in the derivation of the stiffness matrix are chosen, that is $\mathbf{N}_v^e = \mathbf{N}^e$ is called the consistent mass matrix (CMM). It is denoted here by $\mathbf{M}^e{}_C$

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MASS MATRIX CONSTRUCTION

- Variants result according to how the component matrices $\mathbf{M}$ are chosen, and how the parameter μ is determined. The best known scheme of this nature results on taking a weighted average of the consistent and diagonally-lumped mass matrices:
- $$\mathbf{M}^e_{LC} \stackrel{\text{def}}{=} (1-\mu)\mathbf{M}^e_C + \mu\mathbf{M}^e_L$$
- This is called the "lumped-consistent" weighted mass matrix (LC). It is known (since the early 1970s) that the best choice with respect to minimizing low frequency dispersion is $\mu = 1/2$

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MASS MATRIX CONSTRUCTION

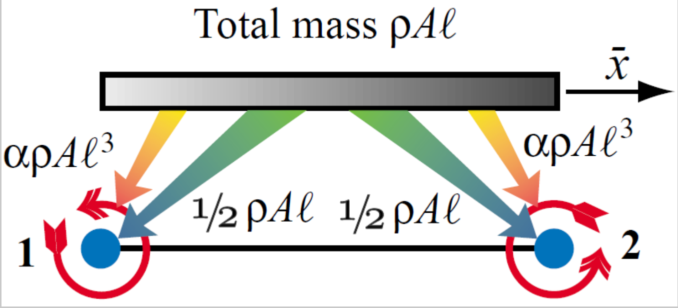
- **The Bernoulli-Euler 2D Beam**

  Consistent mass matrix for Bernoulli-Euler beam:

$$\bar{\mathbf{M}}_C^e = \rho A \int_{-1}^{1} (1/2\ell)(\mathbf{N}^e)^T \mathbf{N}^e \, d\xi = \frac{\rho A \ell}{420} \begin{bmatrix} 156 & 22\ell & 54 & -13\ell \\ 22\ell & 4\ell^2 & 13\ell & -3\ell^2 \\ 54 & 13\ell & 156 & -22\ell \\ -13\ell & -3\ell^2 & -22\ell & 4\ell^2 \end{bmatrix}$$

- Lumped mass matrix for Bernoulli-Euler beam:

$$\bar{\mathbf{M}}_L^e = \rho A \ell \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & \alpha\ell^2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & \alpha\ell^2 \end{bmatrix}$$



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MASS DIAGONALIZATION METHODS

- The construction of consistent mass matrix (CMM) is fully defined by the choice of kinetic energy functional and shape functions. No procedural deviation is possible. On the other hand the construction of a diagonally lumped mass matrix (DLMM) is not a unique process, except for very simple elements in which the lumping is fully defined by conservation and symmetry considerations.
- A consequence of this ambiguity is that various methods have been proposed in the literature, ranging from heuristic through more scientific.

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MASS DIAGONALIZATION METHODS

- **HRZ Lumping (Hinton-Rock-Zienkiewicz, 1976)**

1. For each coordinate direction, select the DOF that contribute to motion in that direction. From this set, separate translational DOF and rotational DOF subsets.
2. Add up the CMM diagonal entries pertaining to the translational DOF subset only. Call the sum S.
3. Apportion $\mathbf{M}^e$ to DLMM entries of both subsets on dividing the CMM diagonal entries by S.
4. Repeat for all coordinate directions.

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MASS DIAGONALIZATION METHODS

- **Lobatto Lumping**
- If the element is one-dimensional and has only translational DOF, can be placed in correspondence with the so-called *Lobatto quadrature* in numerical analysis.
- A Lobatto rule is a $1\mathrm{D}$ Gaussian quadrature formula in which the endpoints of the interval $\xi \in [-1, 1]$ are sample points. If the formula has $p \geq 2$ abscissas, only $p-2$ of those are free. Abscissas are symmetric about the origin $\xi = 0$ and all weights are positive.
- The general form is:

$$\int_{-1}^{1} f(\xi)\, d\xi = w_1 f(-1) + w_p f(1) + \sum_{i=2}^{p-1} w_i f(\xi_i)$$

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MASS DIAGONALIZATION METHODS

- **Lobatto Lumping**
- As a generalization to multiple dimensions, for conciseness we call "FEM Lobatto quadrature" one in which the connected nodes of an element are sample points of an integration rule. But the method runs into some difficulties:
- *Zero or Negative Masses*. If one insists in higher order accuracy, the weights of 2D and 3D Lobatto rules are not necessarily positive. See for example the case of the 6-node triangle in the Exercises. This shortcoming can be alleviated, however, by accepting lower accuracy, or by sticking to product rules in geometries that permit them.

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MASS DIAGONALIZATION METHODS

- **Lobatto Lumping**
- *Rotational Freedoms.* If the element has rotational DOF, Lobatto rules do not exist. Any attempt to transform rules to node rotations inevitably leads to coupling.
- *Varying Properties.* If the element is nonhomogeneous or has varying properties (for instance, a tapered bar element) the construction of Lobatto rules runs into difficulties, since the problem effectively becomes the construction of a quadrature formula with non-unity kernel.

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